MST121 : Supplementary Resource Material for Chapter C2

Question 1

Evaluate each of the following by using the Sum and Constant Multiple Rules in conjunction with a table of standard integrals:

(a) $\int \left(\frac{3}{x} - 4x^2 \right) dx$ (b) $\int (\cos 2x - 5\sin 3x)\, dx$ (c) $\int \left(\frac{4}{x} - 2 - \frac{1}{2} x^2 \right) dx$

(d) $\int \left(e^{2x} - e^{-3x} \right) dx$ (e) $\int \left(\frac{1}{3} e^{-x/5} - \frac{2}{5} e^{x/2} \right) dx$ (f) $\int \left(-\frac{1}{3x} - \exp(-2x) \right) dx$

Question 2

Evaluate the following integrals by firstly expanding each integrand:

(a) $\int (3x-7)^2\, dx$ (b) $\int \frac{x^3 - 1}{2x}\, dx$ (c) $\int \frac{1}{2} e^{5-2x}\, dx$ (d) $\int \sqrt[5]{x} \left(3x^3 - \frac{9}{x^5} \right) dx$

(e) $\int \left(3 - e^{\frac{1}{2}x} \right) \left(3 + e^{\frac{1}{2}x} \right) dx$ (f) $\int \frac{(9-2x)^2}{3x}\, dx$

Question 3

(a) Use the identity $\cos^2 x \equiv \frac{1}{2}(1 + \cos 2x)$ to express $(5 + \cos x)^2$ in terms of $\cos x$.

(b) Hence, or otherwise, determine $\int (5 + \cos x)^2\, dx$.

Question 4

(a) Use the identity $\sin^2 x \equiv \frac{1}{2}(1 - \cos 2x)$ to express $(2 - 3\sin x)^2$ in terms of $\sin x$.

(b) Hence, or otherwise, determine $\int (2 - 3\sin x)^2\, dx$.

Question 5

(a) Show that $(\cos x + \sec x)^2 \equiv \cos^2 x + 2 + \sec^2 x$.

(b) Hence, using the identity $\cos^2 x \equiv \frac{1}{2}(1 + \cos 2x)$ and the result from Activity 2.4 (b), find $\int (\cos x + \sec x)^2\, dx$.

Question 6

(a) Show that $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$ by applying the Quotient Rule for differentiation introduced in Chapter C1.

(b) Hence write down $\int \operatorname{cosec}^2 x dx$

Question 7

Use the identity $\sin^2 x \equiv \frac{1}{2}(1-\cos 2x)$ together with your answer to question 6 to find $\int (\sin x + \operatorname{cosec} x)^2 \, dx$.

Question 8

By dividing $\cos^2 x + \sin^2 x \equiv 1$ through by $\sin^2 x$, use a trigonometric identity together with your answer to question 6 to find $\int \cot^2 x dx$.

Question 9

Use an appropriate trigonometric identity to find $\int \sin^2 6q \, dq$.

Question 10

Find each of the following indefinite integrals:

(a) $\int \sin^4 x \cos x dx$ (b) $\int \frac{x^3}{x^4 - 1} dx$ (c) $\int 2x(x^2 - 11)^{10} dx$

(d) $\int \frac{5x^2}{x^3 - 4} dx$ (e) $\int \tan x \sec^2 x dx$ (f) $\int \sinh^4 x \cosh x dx$

Question 11

A hot air balloonist flying over the Atlantic wishes to calculate his height above sea-level. To allow him to make these calculations the balloonist takes a spherical ball bearing and gently drops it over the side into the ocean below. He finds that it takes 20 seconds for the bearing to fall to sea-level.

(a) In an initial model of the problem we may neglect the force of air resistance and model the ball bearing as a particle falling freely under gravity alone.

Taking the acceleration due to gravity at the Earth's surface to be 10 ms^{-2}, find an upper bound for the height of the balloon above sea-level. Explain clearly why your answer is necessarily an upper bound for the true height.

(b) Explain why including the effects of air resistance will provide a more realistic model.

(c) If we take the effects of air resistance into consideration, the acceleration of the ball bearing is given by $a = 10e^{-0.085t}$. Integrate this expression to find the velocity of the stone at time t seconds, and deduce the terminal velocity.

(d) Integrate the velocity function that you obtained in (c) to obtain an expression for the displacement of the stone at time t seconds.

(e) Hence determine an improved estimate for the true height of the balloon above sea-level, giving your final answer to an appropriate degree of accuracy.

Question 12

Determine the **exact** values each of the following definite integrals:

(a) $\int_1^3 \left(\frac{4}{x} - \frac{5}{x^2} - 2 \right) dx$ (b) $\int_0^{½} 3x^2 \left(x^3 - 1 \right)^4 dx$ (c) $\int_{\frac{p}{6}}^{\frac{p}{4}} \cos x \sin^3 x dx$

Question 13

Draw a sketch of each of the following graphs and use definite integration to determine each of the specified areas:

(a) $y = x^2 - 4x + 5$ between $x = 0$ and $x = 5$

(b) $y = x^3 - 7x^2 + 12x$ between $x = 0$ and $x = 4$

(c) $y = \cos x$ between $x = 0$ and $x = \frac{3p}{2}$

(d) $y = \cos^2 x$ between $x = 0$ and $x = 2p$

(e) $y = e^{2x}$ between $x = 0$ and $x = 1$

(f) $y = \sin x$ between $x = -p$ and $x = p$

(g) $y = \frac{1}{x}$ between $x = 1$ and $x = a$. Comment on your results.

Question 14

(a) Determine the area of the finite region enclosed by the graphs of $y = 1 - x^2$ and $y = x^2 - 1$.

(b) Find the area of the finite region enclosed by the graphs of $y = x^2$ and $y^2 = x$.

Solutions:

Question 1

(a) $$\int\left(\frac{3}{x}-4x^2\right)dx = 3\ln x - 4\times\frac{x^3}{3}+c = 3\ln x-\frac{4}{3}x^3+c$$

(b) $$\int(\cos 2x - 5\sin 3x)\,dx = \frac{1}{2}\sin 2x - 5\times -\frac{1}{3}\cos 3x + c = \frac{1}{2}\sin 2x + \frac{5}{3}\cos 3x + c$$

(c) $$\int\left(\frac{4}{x}-2-\frac{1}{2}x^2\right)dx = 4\ln x - 2x - \frac{1}{2}\times\frac{x^3}{3}+c = 4\ln x - 2x - \frac{1}{6}x^3 + c$$

(d) $$\int\left(e^{2x}-e^{-3x}\right)dx = \frac{1}{2}e^{2x} - \left(-\frac{1}{3}e^{-3x}\right)+c = \frac{1}{2}e^{2x}+\frac{1}{3}e^{-3x}+c$$

(e) $$\int\left(\frac{1}{3}e^{-x/5}-\frac{2}{5}e^{x/2}\right)dx = \frac{1}{3}\times\frac{1}{-1/5}\,e^{-x/5}-\frac{2}{5}\times\frac{1}{1/2}e^{x/2}+c = -\frac{5}{3}e^{-x/5}-\frac{4}{5}e^{x/2}+c$$

(f) $$\int\left(-\frac{1}{3x}-\exp(-2x)\right) = -\frac{1}{3}\times\ln x - \frac{1}{-2}\exp(-2x)+c = -\frac{1}{3}\ln x + \frac{1}{2}\exp(-2x)+c$$

Question 2

(a) As $(3x-7)^2 = 9x^2-42x+49$, it follows that

$$\int(3x-7)^2dx = \int\left(9x^2-42x+49\right)dx = 9\times\frac{x^3}{3}-42\times\frac{x^2}{2}+49x+c$$

$$= 3x^3-21x^2+49x+c$$

(b) As $\frac{x^3-1}{2x}=\frac{x^3}{2x}-\frac{1}{2x}=\frac{1}{2}x^2-\frac{1}{2}\cdot\frac{1}{x}$, it follows that

$$\int\frac{x^3-1}{2x}dx = \int\left(\frac{1}{2}x^2-\frac{1}{2}\cdot\frac{1}{x}\right)dx = \frac{1}{2}\times\frac{x^3}{3}-\frac{1}{2}\times\ln x + c = \frac{1}{6}x^3-\frac{1}{2}\ln x + c$$

(c) As $\frac{1}{2}e^{5-2x}=\frac{1}{2}e^5\times e^{-2x}$, it follows that

$$\int\frac{1}{2}e^{5-2x}dx = \int\left(\frac{1}{2}e^5\times e^{-2x}\right)dx = \frac{1}{2}e^5\times\frac{1}{-2}e^{-2x}+c = -\frac{1}{4}e^{5-2x}+c$$

(d) As $\sqrt[5]{x}\left(3x^3-\frac{9}{x^5}\right)=x^{1/5}\cdot 3x^3-x^{1/5}\cdot 9x^{-5}=3x^{16/5}-9x^{-24/5}$, it follows that

$$\int \sqrt[5]{x}\left(3x^3-\frac{9}{x^5}\right)dx=\int\left(3x^{16/5}-9x^{-24/5}\right)dx=3\times\frac{x^{21/5}}{21/5}-9\times\frac{x^{-19/5}}{-19/5}+c$$

$$=\frac{5}{7}x^{21/5}+\frac{45}{19}x^{-19/5}+c$$

(e) As $\left(3-e^{\frac{1}{2}x}\right)\left(3+e^{\frac{1}{2}x}\right)=9-e^x$ (difference of two squares), we have

$$\int\left(3-e^{\frac{1}{2}x}\right)\left(3+e^{\frac{1}{2}x}\right)dx=\int\left(9-e^x\right)dx=9x-e^x+c$$

(f) As $\frac{(9-2x)^2}{3x}=\frac{81-36x+4x^2}{3x}=\frac{27}{x}-12+\frac{4x}{3}$, we have

$$\int\frac{(9-2x)^2}{3x}dx=\int\left(\frac{27}{x}-12+\frac{4x}{3}\right)dx=27\times\ln x-12x+\frac{4}{3}\times\frac{x^2}{2}+c$$

$$=27\ln x-12x+\frac{2}{3}x^2+c$$

Question 3

(a) $(5+\cos x)^2=25+10\cos x+\cos^2 x=25+10\cos x+\tfrac{1}{2}(1+\cos 2x)$

$$=\tfrac{51}{2}+10\cos x+\tfrac{1}{2}\cos 2x$$

(b) $\therefore\int(5+\cos x)^2\,dx=\int\left(\tfrac{51}{2}+10\cos x+\tfrac{1}{2}\cos 2x\right)dx=\frac{51x}{2}+10\sin x+\frac{1}{4}\sin 2x+c$

Question 4

(a) $(2-3\sin x)^2=4-12\sin x+9\sin^2 x=4-12\sin x+\tfrac{9}{2}(1-\cos 2x)$

$$=\tfrac{17}{2}-12\sin x-\tfrac{9}{2}\cos 2x$$

(b) $\therefore\int(2-3\sin x)^2\,dx=\int\left(\tfrac{17}{2}-12\sin x-\tfrac{9}{2}\cos 2x\right)dx=\tfrac{17}{2}x+12\cos x-\tfrac{9}{4}\sin 2x+c$

Question 5

(a) $(\cos x + \sec x)^2 = \cos^2 x + 2\cos x \sec x + \sec^2 x = \cos^2 x + 2 + \sec^2 x$

(b) $\therefore (\cos x + \sec x)^2 = \cos^2 x + 2 + \sec^2 x = \tfrac{1}{2}(1 + \cos 2x) + 2 + \sec^2 x$

$$= \tfrac{5}{2} + \tfrac{1}{2}\cos 2x + \sec^2 x$$

$$\therefore \int (\cos x + \sec x)^2 dx = \int \left(\tfrac{5}{2} + \tfrac{1}{2}\cos 2x + \sec^2 x \right) dx = \tfrac{5}{2}x + \tfrac{1}{4}\sin 2x + \tan x + c$$

Question 6

(a) As $\cot x = \dfrac{\cos x}{\sin x}$, informal application of the quotient rule gives

$$\frac{d}{dx}\left(\frac{\cos x}{\sin x} \right) = \frac{\sin x \cdot (-\sin x) - \cos x \cdot \cos x}{\sin^2 x} = -\frac{\sin^2 x + \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x}$$

$$= -\operatorname{cosec}^2 x$$

(b) $\therefore \int \operatorname{cosec}^2 x\, dx = -\cot x + c$

Question 7

$$(\sin x + \operatorname{cosec} x)^2 = \sin^2 x + 2\sin x \operatorname{cosec} x + \operatorname{cosec}^2 x = \sin^2 x + 2 + \operatorname{cosec}^2 x$$

$$\therefore (\sin x + \operatorname{cosec} x)^2 = \sin^2 x + 2 + \operatorname{cosec}^2 x = \tfrac{1}{2}(1 - \cos 2x) + 2 + \operatorname{cosec}^2 x$$

$$= \tfrac{5}{2} - \tfrac{1}{2}\cos 2x + \operatorname{cosec}^2 x$$

$$\therefore \int (\sin + \operatorname{cosec} x)^2 dx = \int \left(\tfrac{5}{2} - \tfrac{1}{2}\cos 2x + \operatorname{cosec}^2 x \right) dx = \tfrac{5}{2}x - \tfrac{1}{4}\sin 2x - \cot x + c$$

Question 8

Dividing $\cos^2 x + \sin^2 x \equiv 1$ through by $\sin^2 x$ gives $\cot^2 x + 1 \equiv \operatorname{cosec}^2 x$.

$$\therefore \int \cot^2 x\, dx = \int \left(\operatorname{cosec}^2 x - 1 \right) dx = -\cot x - x + c$$

Question 9

Applying $\sin^2 x \equiv \tfrac{1}{2}(1 - \cos 2x)$ gives $\sin^2 6q = \tfrac{1}{2}(1 - \cos 12q)$

$$\therefore \int \sin^2 6q\, dq = \int \tfrac{1}{2}(1 - \cos 12q)\, dq = \frac{1}{2}\left(q - \frac{1}{12}\sin 12q \right) + c$$

Question 10

(a) $\int \sin^4 x \cos x dx$ is of the form $\int (f(x))^n f'(x) dx$, where $f(x) = \sin x$ and $n = 4$

Hence, by equation (2.3) on page 23 of Chapter C2,

$$\int \sin^4 x \cos x dx = \frac{1}{5}\sin^5 x + c.$$

(b) $\int \frac{x^3}{x^4 - 1} dx$ can be re-written as $\frac{1}{4}\int \frac{4x^3}{x^4 - 1} dx$

Now as $\int \frac{4x^3}{x^4 - 1} dx$ is of the form $\int \frac{f'(x)}{f(x)} dx$, equation (2.4) on page 23 of Chapter C2 tells us that it evaluates to $\ln\left(x^4 - 1\right) + c$.

$$\therefore \int \frac{x^3}{x^4 - 1} dx = \frac{1}{4}\ln\left(x^4 - 1\right) + c$$

(c) $\int 2x\left(x^2 - 11\right)^{10} dx$ is of the form $\int (f(x))^n f'(x) dx$ where $f(x) = x^2 - 11$ and $n = 10$.

Hence, by equation (2.3) on page 23 of Chapter C2,

$$\int 2x\left(x^2 - 11\right)^{10} dx = \frac{1}{11}\left(x^2 - 11\right)^{11} + c$$

(d) $\int \frac{5x^2}{x^3 - 4} dx$ can be re-written as $\frac{5}{3}\int \frac{3x^2}{x^3 - 4} dx$.

Now as $\int \frac{3x^2}{x^3 - 4} dx$ is of the form $\int \frac{f'(x)}{f(x)} dx$, equation (2.4) on page 23 of Chapter C2 tells us that it evaluates to $\ln\left(x^3 - 4\right) + c$.

$$\therefore \int \frac{5x^2}{x^3 - 4} dx = \frac{5}{3}\ln\left(x^3 - 4\right) + c$$

(e) $\int \tan x \sec^2 x dx$ is of the form $\int (f(x))^n f'(x) dx$ where $f(x) = \tan x$ and $n = 1$.

Hence, by equation (2.3) on page 23 of C2, $\int \tan x \sec^2 x dx = \frac{1}{2}\tan^2 x + c$

(f) $\int \sinh^4 x \cosh x dx$ is of the form $\int (f(x))^n f'(x) dx$ where $f(x) = \sinh x$ and $n = 4$.

Hence, by equation (2.3) on page 23 of Chapter C2,

$$\int \sinh^4 x \cosh x dx = \frac{1}{5} \sinh^5 x + c$$

Question 11

(a) As the acceleration is constant, we can model the motion by the equation $s = \tfrac{1}{2} a t^2 + v_0 t + s_0$.

Substituting $a = 10,\ t = 20,\ v_0 = s_0 = 0$ into this equation of motion gives $s = \tfrac{1}{2} \times 10 \times 20^2 = 2000$ metres.

This is an upper bound because we have neglected the force of air resistance which would have reduced the acceleration and slowed the descent of the ball bearing.

(b) As the speed increases, so does the air resistance.

(c) The velocity is given by $v = \int a dt$

$$\therefore v = \int 10 e^{-0.085t} dt = \frac{10}{-0.085} e^{-0.085t} + c_1 = -\frac{2000}{17} e^{-0.085t} + c_1$$

Now as $v = 0$ when $t = 0$, it follows that $c_1 = \frac{2000}{17}$.

Hence $v = \frac{2000}{17} - \frac{2000}{17} e^{-0.085t} = \frac{2000}{17}\left(1 - e^{-0.085t}\right)$

The terminal velocity is given by $\lim_{t \to \infty} v(t) = \frac{2000}{17}$ metres per second.

(d) The displacement, measured vertically downwards from the balloon, is given by $s = \int v dt$.

$$\therefore s = \frac{2000}{17} \int \left(1 - e^{-0.085t}\right) dt = \frac{2000}{17}\left(t + \frac{200}{17} e^{-0.085t}\right) + c_2$$

Now as $s = 0$ when $t = 0$, it follows that $c_2 = -\frac{400000}{289}$.

$$\therefore s=\frac{2000}{17}\left(t+\frac{200}{17}e^{-0.085t}\right)-\frac{400000}{289}$$

(e) $$\therefore s(20)=\frac{2000}{17}\left(20+\frac{200}{17}e^{-0.085\times 20}\right)-\frac{400000}{289}=1221.707\ldots\approx 1200 \text{ metres.}$$

Question 12

(a) $$\int_1^3\left(\frac{4}{x}-\frac{5}{x^2}-2\right)dx=\left[4\ln|x|+\frac{5}{x}-2x\right]_1^3$$

$$=\left(4\ln 3+\frac{5}{3}-6\right)-\left(4\ln 1+\frac{5}{1}-2\right)$$

$$=4\ln 3-\frac{22}{3}$$

(b) $\int 3x^2\left(x^3-1\right)^4dx=$ is of the form $\int\left(f(x)\right)^n f'(x)dx$ where $f(x)=x^3-1$ and $n=4$.

Hence, by equation (2.3) on page 23 of Chapter C2,

$$\int 3x^2\left(x^3-1\right)^4dx=\tfrac{1}{5}\left(x^3-1\right)^5+c$$

$$\int_0^{½}\left(x^3-1\right)^4dx=\left[\tfrac{1}{5}\left(x^3-1\right)^5\right]_0^{½}=-\frac{16807}{163840}-\left(-\frac{1}{5}\right)=\frac{15961}{163840}\quad(\approx 0.0974).$$

(c) $\int\cos x\sin^3 x\,dx$ is of the form $\int\left(f(x)\right)^n f'(x)dx$ where $f(x)=\sin x$ and $n=3$.

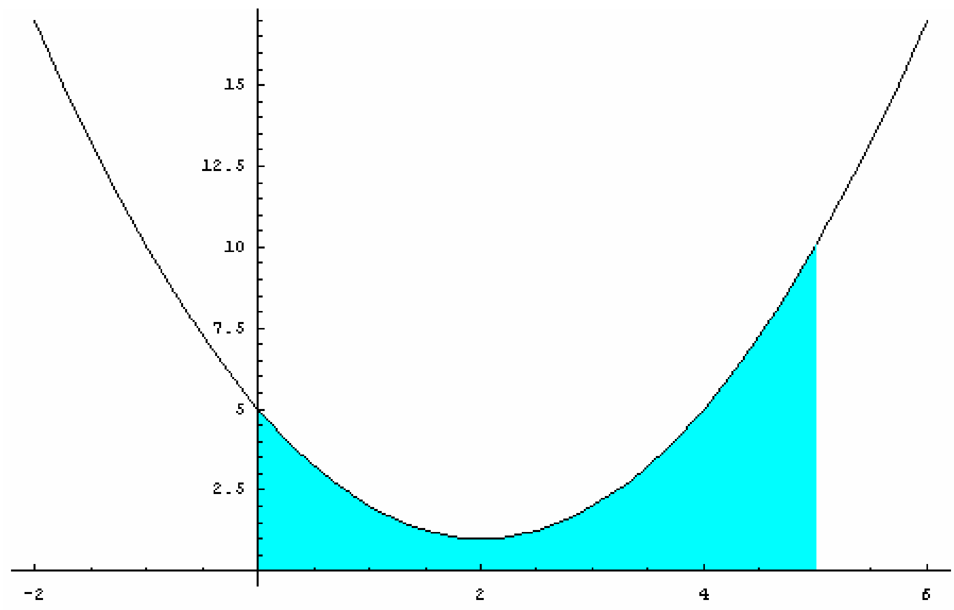
Hence, by equation (2.3) on page 23 of Chapter C2,

$$\int\cos x\sin^3 x\,dx=\tfrac{1}{4}\sin^4 x+c$$

$$\therefore\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\cos x\sin^3 x\,dx=\left[\tfrac{1}{4}\sin^4 x\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}=\frac{1}{4}\left(\left(\frac{1}{\sqrt{2}}\right)^4-\left(\frac{1}{2}\right)^4\right)=\frac{3}{64}$$

Question 13

(a) Expressing $x^2 - 4x + 5$ in the completed square form $(x-2)^2 + 1$ shows that the graph of $y = x^2 - 4x + 5$ is obtained from that of $y = x^2$ by a translation 2 units to the right and 1 unit up. Hence its graph is as shown below.

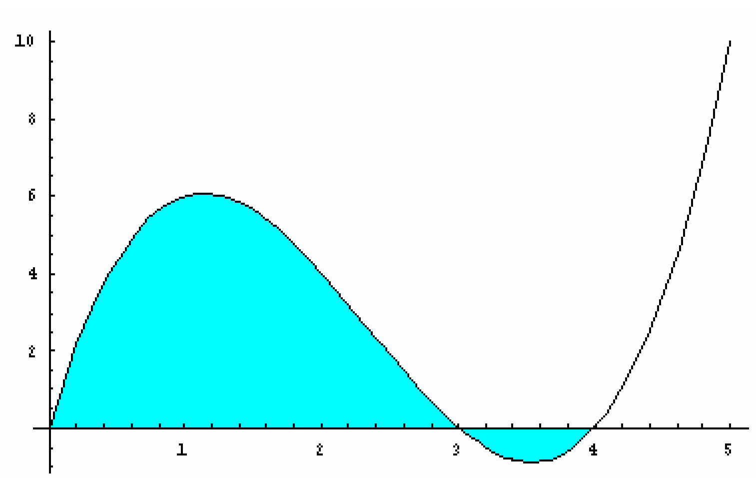


As the area between $x = 0$ and $x = 5$ lies wholly above the x-axis, it is given by the value of the definite integral $\int_0^5 (x^2 - 4x + 5)\,dx$.

$$\therefore A = \int_0^5 (x^2 - 4x + 5)\,dx = \left[\tfrac{1}{3}x^3 - 2x^2 + 5x\right]_0^5 = \tfrac{50}{3} - 0 = \tfrac{50}{3}$$

(b) $x^3 - 7x^2 + 12x = x(x^2 - 7x + 12) = x(x-3)(x-4)$

Hence the curve is a ‘right way-up’ cubic that intersects the x-axis at $x = 0$, $x = 3$ and $x = 4$.



To evaluate the area between $x=0$ and $x=4$, we must take into consideration that the curve is below the x-axis between $x=3$ and $x=4$. Consequently the definite integral for the region between $x=3$ and $x=4$ will return a negative value and we must allow for this when computing the area. Hence

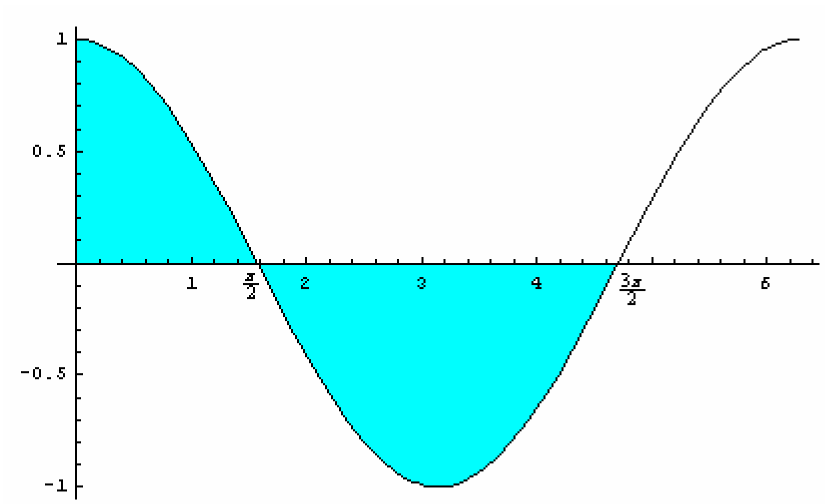
$$A=\int_0^3\left(x^3-7x^2+12x\right)dx-\int_3^4\left(x^3-7x^2+12x\right)dx$$

$$=\left[\frac{1}{4}x^4-\frac{7}{3}x^3+6x^2\right]_0^3-\left[\frac{1}{4}x^4-\frac{7}{3}x^3+6x^2\right]_3^4$$

$$=\frac{45}{4}-\left(\frac{32}{3}-\frac{45}{4}\right)$$

$$=\frac{71}{6}$$

(c) Part of the graph of $y=\cos x$ between $x=0$ and $x=\frac{3p}{2}$ is also below the x-axis, as the following graph shows:

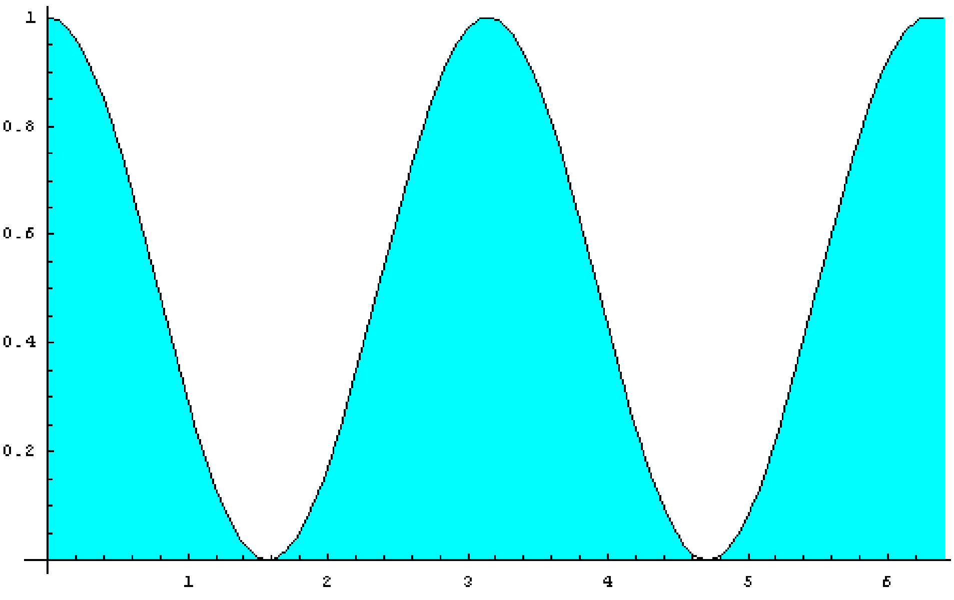


By symmetry, the area is given by

$$A=3\int_0^{p/2}\cos x\,dx=3\left[\sin x\right]_0^{p/2}=3\left(\sin\tfrac{p}{2}-\sin 0\right)=3$$

(d) The graph of $y=\cos^2 x$ is most easily visualised by comparing it with that of $y=\cos x$ (to which it is similar), but with all the negative sections reflected in the x-axis.

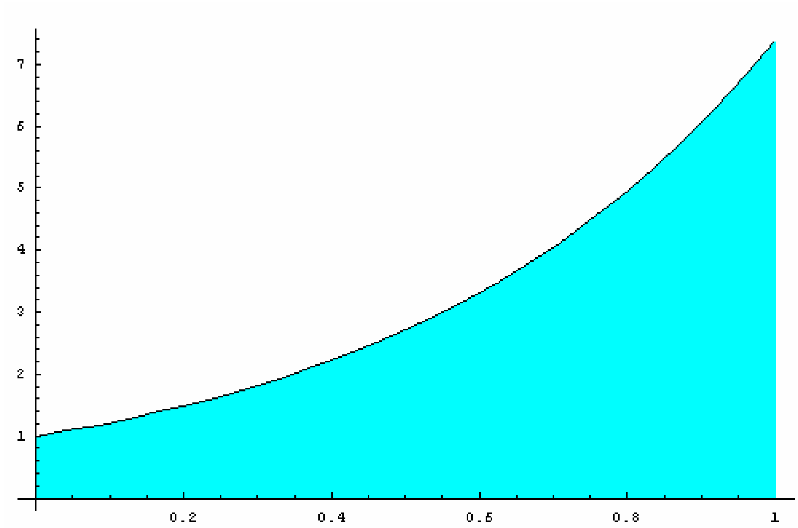
Hence all the curve lies above the x-axis, and so the area between $x=0$ and $x=2p$ is given by $A=\int_0^{2p}\cos^2 x\,dx$.



Using the result $\cos^2 x \equiv \frac{1}{2}(1+\cos 2x)$ introduced in question three, we have

$$A=\int_0^{2p} \cos^2 x dx = \tfrac{1}{2}\int_0^{2p} (1+\cos 2x)\,dx = \tfrac{1}{2}\left[x+\tfrac{1}{2}\sin 2x\right]_0^{2p} = \tfrac{1}{2}(2p-0)=p$$

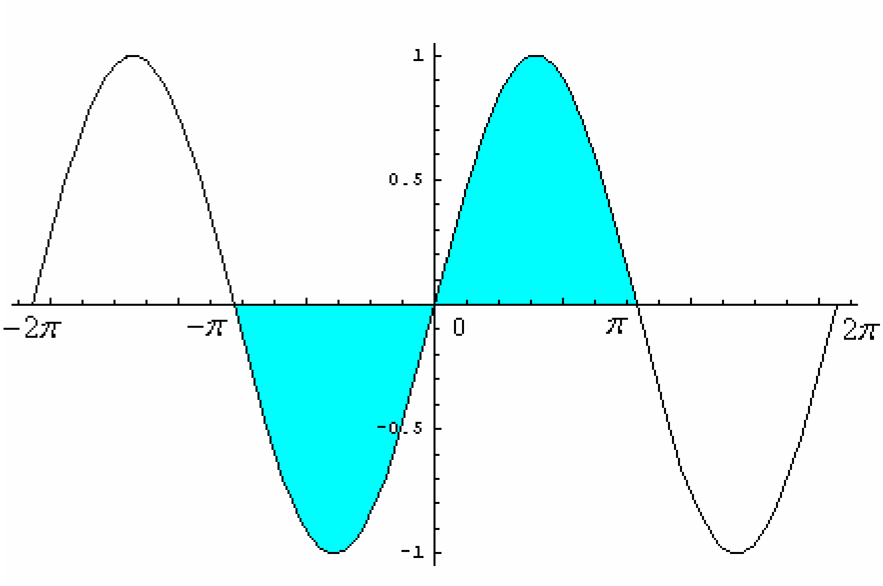
(e) The graph of $y=e^{2x}$ should be quite familiar by now, and is presented below without justification.



As all the curve lies above the x-axis, the area between $x=0$ and $x=1$ is given by $A=\int_0^1 e^{2x}\,dx$.

$$A=\int_0^1 e^{2x}\,dx = \left[\tfrac{1}{2}e^{2x}\right]_0^1 = \tfrac{1}{2}\left(e^2-e^0\right)=\tfrac{1}{2}\left(e^2-1\right)$$

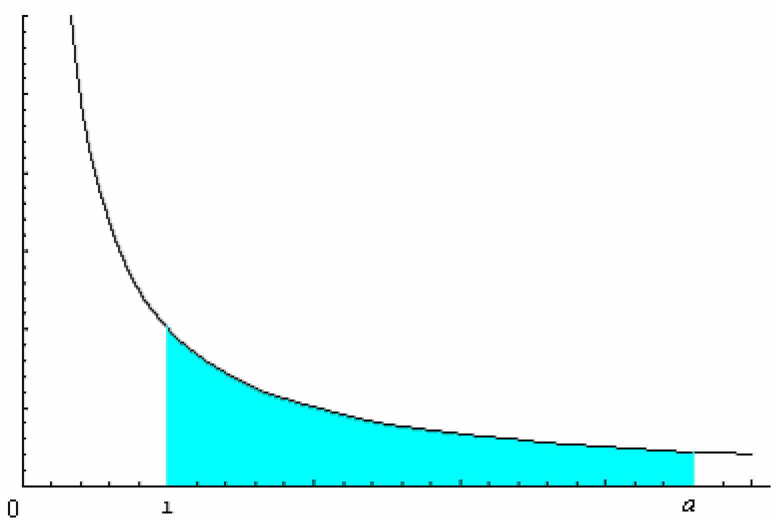
(f) The graph of $y = \sin x$ should also be quite familiar by now.



As $y = \sin x$ has rotational symmetry of order two about the origin, we can appeal to symmetry to give

$$A = 2\int_0^p \sin x dx = -2[\cos x]_0^p = -2(-1-1) = 4$$

(g) The graph of $y = \frac{1}{x}$ in the first quadrant is given below.



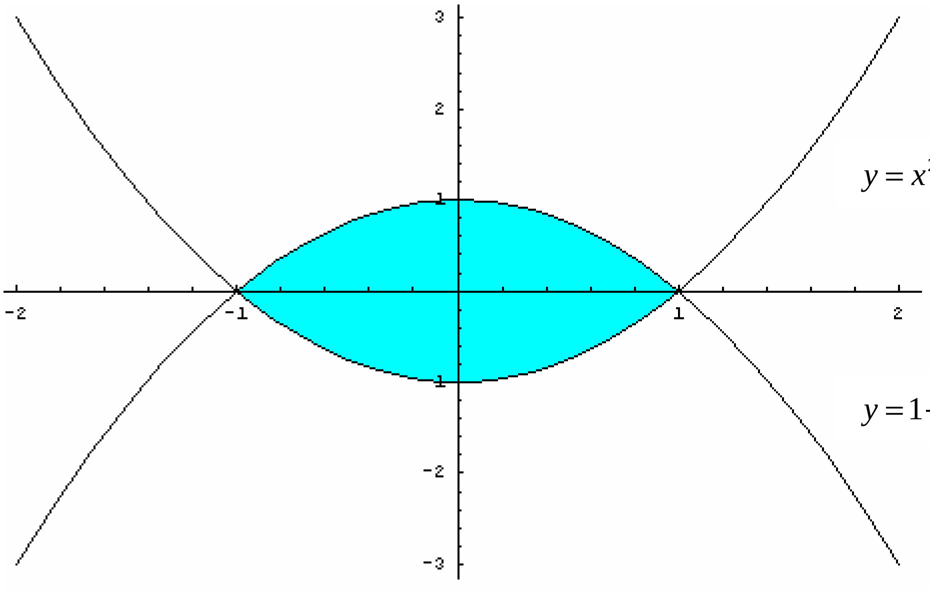
As the curve lies wholly above the x-axis, the area is given by

$$A = \int_1^a \frac{1}{x} dx = [\ln x]_1^a = \ln a - \ln 1 = \ln a$$

For $y = \frac{1}{x}$, the area between $x = 1$ and any number greater than 1 is always given by the natural logarithm of the greater number.

Question 14

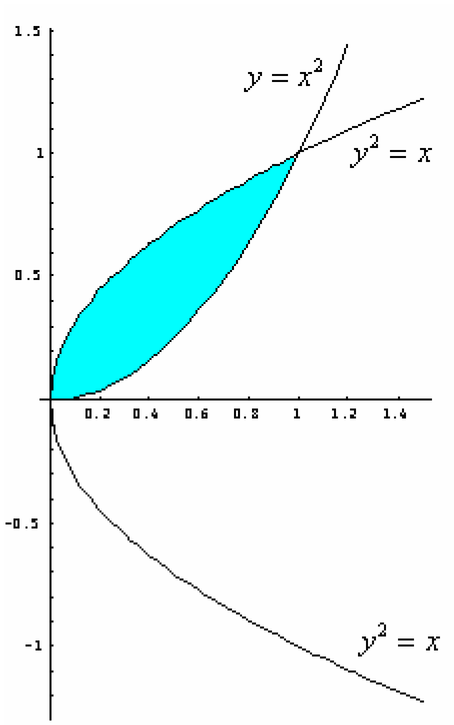
(a) The graphs of $y = 1 - x^2$ and $y = x^2 - 1$ are given below.



By symmetry, the area enclosed by the graphs is given by

$$A = 4 \cdot \int_0^1 (1 - x^2)\, dx = 4 \cdot \left[x - \frac{1}{3}x^3 \right]_0^1 = 4\left(\frac{2}{3} - 0\right) = \frac{8}{3}$$

(b)



The graphs clearly show that the curves intersect at $x = 1$.

Hence the area enclosed by the curves is given by

$$A = \int_0^1 \sqrt{x}\, dx - \int_0^1 x^2\, dx$$

$$= \left[\frac{2}{3}x^{3/2}\right]_0^1 - \left[\frac{1}{3}x^3\right]_0^1$$

$$= \frac{2}{3} - \frac{1}{3}$$

$$= \frac{1}{3}$$