MST121 : Supplementary Resource Material for Chapter C1

Question 1

Find the gradient of the graph of the function $f(x) = 2x^2 - 9x + 11$ at the point with co-ordinates $(3,2)$.

Question 2

(a) Use the definition $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ to find the derived function of $f(x) = \dfrac{1}{x^2}$.

(b) Confirm that your answer to (a) is correct by differentiating $f(x)$ using result (1.4) for power functions (page 16 of C1).

Question 3

By firstly expressing each function in index notation, use result (1.4) for power functions (page 16 of C1) to differentiate each of the following:

$(a)\ \ f(x) = \sqrt[3]{x^2} \quad (b)\ \ f(x) = \dfrac{2}{\sqrt{x}} \quad (c)\ \ f(x) = \dfrac{1}{\sqrt[4]{x^3}} \quad (d)\ \ f(x) = \dfrac{x}{\sqrt[3]{x^2}}$

Question 4

A motorbike accelerating along a race track travels a distance $s = \frac{1}{2}t^2 + 7t + 6$ metres in the first 60 seconds of the race.

(a) Find the velocity, v ms^{-1}, of the motorbike at time t seconds, where $0 \le t \le 60$.

(b) What is the velocity of the motorbike after 30 seconds?

(c) At what time does the motorbike have a velocity of 18 ms^{-1}?

(d) Explain why the acceleration of the motorbike is constant, and write down its value and the units that it is measured in.

Question 5

Determine the gradient of the graph of the polynomial function $f(x) = 7x^4 - 19x^3 + 21x^2 - 11x - 15$ at the point with co-ordinates $(5,2455)$.

Question 6

(a) Differentiate the cubic polynomial $p(x) = 2x^3 + 3x^2 - 72x + 14$.

(b) Find the co-ordinates of the stationary points.

(c) Classify these stationary points by using the First Derivative Test.

(d) Use the Increasing/Decreasing Criterion to find intervals on which $p(x)$ is increasing or decreasing.

(e) Use the results of parts (b)-(d) to draw a sketch of the polynomial $p(x)$.

Question 7

(a) Differentiate the cubic polynomial $p(x) = 2x^3 + 21x^2 + 60x + 1$.

Find the co-ordinates of the stationary points and classify these by using the Second Derivative Test. Hence sketch the graph of $y = p(x)$.

(b) Deduce the greatest and least values of $p(x)$ on the interval $I = [-6, 0]$.

Question 8

Differentiate each of the following functions:

(a) $u(x) = 3\cos x - 5\sin x + 1$

(b) $v(x) = 3e^x - 5\ln x + 2x + 5$

(c) $w(x) = 2x^3 - 4x^2 + 9x - 1 + \frac{4}{x} - \frac{2}{x^2} + \frac{3}{x^3}$

Question 9

Find each of the following derivatives using the notation indicated:

(a) $\frac{dP}{du}$, $\frac{d^2P}{du^2}$ and $\frac{d^3P}{du^3}$ where $P = 3\sin u - 5\cos u$.

(b) $\dot{v}$ and $\ddot{v}$ where $v = 3e^t - t^2 + 2t - 7$

(c) $\left.\frac{dy}{dx}\right|_{x=3}$ and $\left.\frac{d^2y}{dx^2}\right|_{x=5}$ when $y = 2e^x - \ln x$

Question 10

Use the Product Rule to differentiate each of the following functions:

(a) $y = x^2 e^x$ (b) $y = \left(x^2 - 5\right)\ln x$ (c) $y = \cos t \cdot \ln t$

Question 11

Use the Quotient Rule to differentiate each of the following functions:

$$(a)\quad k(t)=\frac{t}{\ln t+1}\qquad (b)\quad h(u)=\frac{e^u+1}{e^u-1}\qquad (c)\quad x(t)=\frac{t^2+1}{t^2-1}$$

Question 12

Use either the Composite Rule or the Chain Rule to differentiate each of the following functions:

$$(a)\quad f(x)=\sin\left(x^2+1\right)\qquad (b)\quad x(t)=\ln\left(e^t+1\right)\qquad (c)\quad y=\sin^4 x$$

Question 13

Differentiate each of the following functions by applying an appropriate rule :

$$(a)\quad y=e^x\ln x\qquad (b)\quad y=\cos\left(x^4\right)\qquad (c)\quad y=\cos^4 x\qquad (d)\quad y=xe^{-x}$$

Question 14

Differentiate each of the following functions by firstly applying the rules of logarithms to simplify their definitions:

$$(a)\quad y=\ln\left(2x^2-1\right)^5\qquad (b)\quad y=\ln\sqrt{\frac{x+1}{x-1}}\qquad (c)\quad y=\ln\left\{\left(x^2-1\right)^7\left(x^2+1\right)^8\right\}$$

Question 15

(a) If $y=\sec x$ show that $\dfrac{dy}{dx}=\sec x\tan x$.

(b) If $y=\cot x$ show that $\dfrac{dy}{dx}=-\operatorname{cosec}^2 x$.

[Hint: Write $\cot x=\dfrac{\cos x}{\sin x}$, and then apply the Quotient Rule].

(c) If $y=\operatorname{cosec} x$ show that $\dfrac{dy}{dx}=-\operatorname{cosec} x\cot x$.

Question 16

Farmer Giles has 2000m of fencing and wishes to build a rectangular sheep pen with a semi-circular end, as shown in the diagram. What should the dimensions of the pen be if it is to enclose the maximum area?

Answers:

Question 1

By result (1.1) on page 11 of *C*1 we know that $f'(x) = 4x - 9$. Hence the gradient at $(3,2)$ is $f'(3) = 4 \times 3 - 9 = 3$.

Question 2

(a) $f(x+h) - f(x) = \dfrac{1}{(x+h)^2} - \dfrac{1}{x^2} = \dfrac{x^2 - (x+h)^2}{x^2(x+h)^2} = \dfrac{-2xh - h^2}{x^2(x+h)^2} = \dfrac{-h(2x+h)}{x^2(x+h)^2}$

$\therefore \dfrac{f(x+h) - f(x)}{h} = \dfrac{-h(2x+h)}{x^2 h(x+h)^2} = \dfrac{-(2x+h)}{x^2(x+h)^2}$

$\therefore f'(x) = \lim_{h \to 0} \left\{ \dfrac{-(2x+h)}{x^2(x+h)^2} \right\} = -\dfrac{2x}{x^2 \cdot x^2} = -\dfrac{2}{x^3}$

(b) As $\dfrac{1}{x^2} = x^{-2}$, application of (1.4) gives $f'(x) = -2x^{-3} = -\dfrac{2}{x^3}$ [as before].

Question 3

(a) With $f(x) = \sqrt[3]{x^2} = x^{2/3}$, we have $f'(x) = \tfrac{2}{3} x^{-1/3}$.

(b) With $f(x) = \dfrac{2}{\sqrt{x}} = 2x^{-1/2}$, we have $f'(x) = -x^{-3/2}$.

(c) With $f(x) = \dfrac{1}{\sqrt[4]{x^3}} = x^{-3/4}$, we have $f'(x) = -\tfrac{3}{4} x^{-7/4}$.

(d) With $f(x) = \dfrac{x}{\sqrt[3]{x^2}} = \dfrac{x}{x^{2/3}} = x^{1/3}$, we have $f'(x) = \tfrac{1}{3} x^{-2/3}$.

Question 4

(a) With $s = \tfrac{1}{2} t^2 + 7t + 6$ we have $v = \dfrac{ds}{dt} = t + 7$.

(b) When $t = 30$, $v = 37$ ms^{-1}.

(c) Solving $t + 7 = 18$ gives $t = 11$ seconds.

(d) The acceleration is given by $a = \dfrac{dv}{dt} = 1$ ms^{-2}. It is constant because it is independent of *t*, and its units are metres per second per second.

Question 5

Differentiating the polynomial $f(x) = 7x^4 - 19x^3 + 21x^2 - 11x - 15$ gives $f'(x) = 28x^3 - 57x^2 + 42x - 11$. The gradient of the graph at the point with co-ordinates $(5,2455)$ is given by $f'(5) = 2274$.

Question 6

(a) Differentiating the cubic polynomial $p(x) = 2x^3 + 3x^2 - 72x + 14$ gives $p'(x) = 6x^2 + 6x - 72 = 6(x^2 + x - 12) = 6(x+4)(x-3)$.

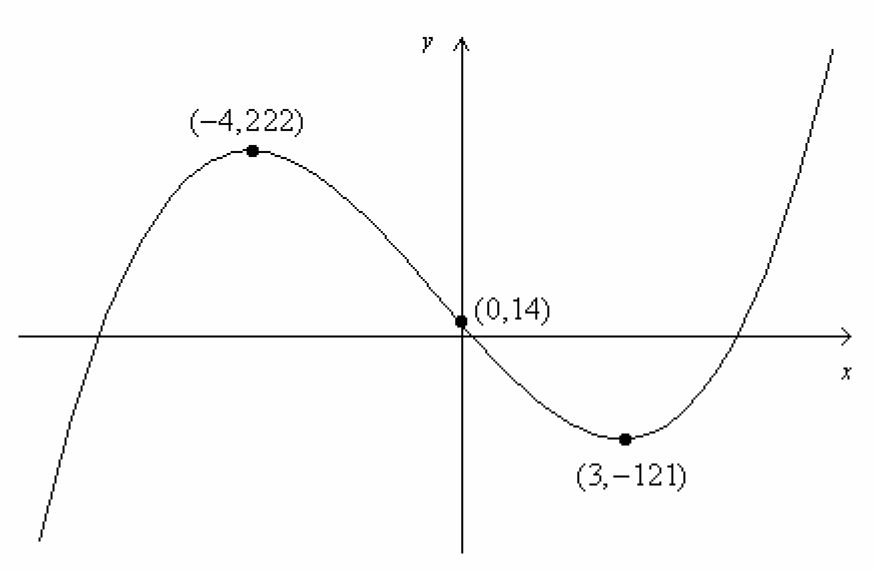
(b) The x co-ordinates of the stationary points are the solutions to $p'(x) = 0$. Hence these are $x = -4$ and $x = 3$. As $p(-4) = 222$ and $p(3) = -121$, it follows that the co-ordinates of the stationary points are $(-4,222)$ and $(3,-121)$.

(c) For $x = -4$ we may take $x_L = -4.01$ and $x_R = -3.99$. Then $f'(x_L) \approx 0.42 > 0$ and $f'(x_R) \approx -0.42 < 0$. Hence, by the First Derivative Test, $(-4,222)$ is a local maximum.

For $x = 3$ we may take $x_L = 2.99$ and $x_R = 3.01$. Then $f'(x_L) \approx -0.42 < 0$ and $f'(x_R) \approx 0.42 > 0$. Hence, by the First Derivative Test, $(3,-121)$ is a local minimum.

(d) By the Increasing/Decreasing Criterion, $p(x)$ is increasing on $(-\infty,-4)$ and $(3,\infty)$, and is decreasing on $(-4,3)$.

(e)



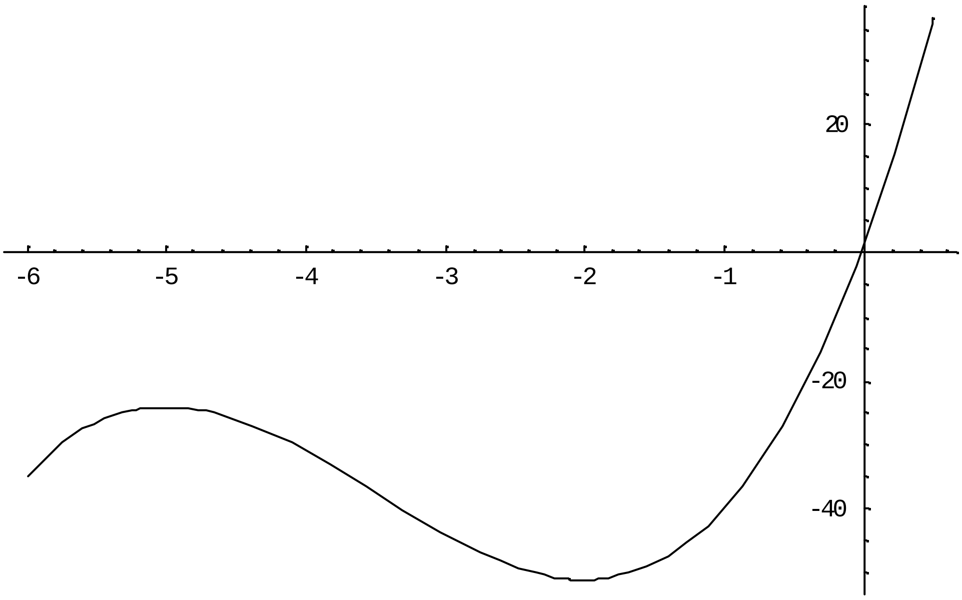
Question 7

(a) Differentiating the cubic polynomial $p(x) = 2x^3 + 21x^2 + 60x + 1$ gives $p'(x) = 6x^2 + 42x + 60 = 6(x^2 + 7x + 10) = 6(x+5)(x+2)$.

The x co-ordinates of the stationary points are the solutions to $p'(x)=0$, which are $x=-5$ and $x=-2$. As $p(-5)=-24$ and $p(-2)=-51$, it follows that the co-ordinates of the stationary points are $(-5,-24)$ and $(-2,-51)$.

Now as $p'(x)=6x^2+42x+60$, it follows that $p''(x)=12x+42$. Hence $p''(-5)=-18<0$ and $p''(-2)=18>0$.

It follows from the Second Derivative Test that $(-5,-24)$ is a local maximum and $(-2,-51)$ is a local minimum.



(b) The graph of $p(x)$ shows that the minimum value of $p(x)$ on the interval $[-6,0]$ occurs at $x=-2$ and the maximum value at $x=0$. Hence the minimum value is -51 and the maximum value is $p(0)=1$.

Question 8

(a) With $u(x)=3\cos x-5\sin x+1$ we have $u'(x)=-3\sin x-5\cos x$.

(b) With $v(x)=3e^x-5\ln x+2x+5$ we have $v'(x)=3e^x-\frac{5}{x}+2$

(c) $w(x)=2x^3-4x^2+9x-1+\frac{4}{x}-\frac{2}{x^2}+\frac{3}{x^3}=2x^3-4x^2+9x-1+4x^{-1}-2x^{-2}+3x^{-3}$

Hence $w'(x)=6x^2-8x+9-4x^{-2}+4x^{-3}-9x^{-4}$

$$=6x^2-8x+9-\frac{4}{x^2}+\frac{4}{x^3}-\frac{9}{x^4}$$

Question 9

(a) As $P = 3\sin u - 5\cos u$ we have $\dfrac{dP}{du} = 3\cos u + 5\sin u$, $\dfrac{d^2P}{du^2} = -3\sin u + 5\cos u$ and $\dfrac{d^3P}{du^3} = -3\cos u - 5\sin u$.

(b) With $v = 3e^t - t^2 + 2t - 7$ we have $\dot{v} = 3e^t - 2t + 2$ and $\ddot{v} = 3e^t - 2$

(c) As $y = 2e^x - \ln x$, we have $\dfrac{dy}{dx} = 2e^x - \dfrac{1}{x}$ and $\dfrac{d^2y}{dx^2} = 2e^x + \dfrac{1}{x^2}$

$\therefore \left.\dfrac{dy}{dx}\right|_{x=3} = 2e^3 - \dfrac{1}{3}$ and $\left.\dfrac{d^2y}{dx^2}\right|_{x=5} = 2e^5 + \dfrac{1}{25}$

Question 10

(a) $y = x^2e^x$ is of the form $y = f(x)g(x)$ where $f(x) = x^2$ and $g(x) = e^x$.

Hence $f'(x) = 2x$ and $g'(x) = e^x$

$\therefore$ Application of the Product Rule $(fg)' = f'g + fg'$ gives

$$\frac{dy}{dx} = 2xe^x + x^2e^x = xe^x(2+x)$$

(b) $y = \left(x^2 - 5\right)\ln x$ is of the form $y = f(x)g(x)$ where $f(x) = x^2 - 5$ and $g(x) = \ln x$.

Hence $f'(x) = 2x$ and $g'(x) = \dfrac{1}{x}$

$\therefore$ Application of the Product Rule $(fg)' = f'g + fg'$ gives

$$\therefore \frac{dy}{dx} = 2x\ln x + \left(x^2 - 5\right)\cdot\frac{1}{x} = 2x\ln x + \frac{x^2 - 5}{x}$$

(c) $y = \cos t \cdot \ln t$ is of the form $y = f(t)g(t)$ where $f(t) = \cos t$ and $g(t) = \ln t$.

Hence $f'(t) = -\sin t$ and $g'(t) = \dfrac{1}{t}$

$\therefore$ Application of the Product Rule $(fg)' = f'g + fg'$ gives

$$\frac{dy}{dt} = -\sin t \cdot \ln t + \cos t \cdot \frac{1}{t} = \frac{\cos t}{t} - \ln t \sin t$$

Question 11

(a) $k(t) = \dfrac{t}{\ln t + 1}$ is of the form $\dfrac{f(t)}{g(t)}$, where $f(t) = t$ and $g(t) = \ln t + 1$

Hence $f'(t) = 1$ and $g'(t) = \dfrac{1}{t}$

$\therefore$ Application of the Quotient Rule $\left(\dfrac{f}{g}\right)' = \dfrac{gf' - fg'}{g^2}$ gives

$$k'(t) = \frac{(\ln t + 1)\cdot 1 - t \cdot \frac{1}{t}}{(\ln t + 1)^2} = \frac{\ln t + 1 - 1}{(\ln t + 1)^2} = \frac{\ln t}{(\ln t + 1)^2}$$

(b) $h(u) = \dfrac{e^u + 1}{e^u - 1}$ is of the form $\dfrac{f(u)}{g(u)}$, where $f(u) = e^u + 1$ and $g(u) = e^u - 1$

Hence $f'(u) = g'(u) = e^u$

$\therefore$ Application of the Quotient Rule $\left(\dfrac{f}{g}\right)' = \dfrac{gf' - fg'}{g^2}$ gives

$$h'(u) = \frac{(e^u - 1)\cdot e^u - (e^u + 1)\cdot e^u}{(e^u - 1)^2} = \frac{e^{2u} - e^u - e^{2u} - e^u}{(e^u - 1)^2} = -\frac{2e^u}{(e^u - 1)^2}$$

(c) $x(t) = \dfrac{t^2 + 1}{t^2 - 1}$ is of the form $\dfrac{f(t)}{g(t)}$, where $f(t) = t^2 + 1$ and $g(t) = t^2 - 1$

Hence $f'(t) = g'(t) = 2t$

$\therefore$ Application of the Quotient Rule $\left(\dfrac{f}{g}\right)' = \dfrac{gf' - fg'}{g^2}$ gives

$$x'(t) = \frac{(t^2 - 1)\cdot 2t - (t^2 + 1)\cdot 2t}{(t^2 - 1)^2} = \frac{2t\left[(t^2 - 1) - (t^2 + 1)\right]}{(t^2 - 1)^2} = -\frac{4t}{(t^2 - 1)^2}$$

Question 12

(a) With $f(x) = \sin\left(x^2+1\right)$ we have

$$f'(x) = \cos\left(x^2+1\right)\cdot\frac{d}{dx}\left(x^2+1\right) = 2x\cos\left(x^2+1\right)$$

(b) With $x(t) = \ln\left(e^t+1\right)$ we have $x'(t) = \frac{1}{e^t+1}\cdot\frac{d}{dt}\left(e^t+1\right) = \frac{e^t}{e^t+1}$

(c) $y = \sin^4 x$ can be re-written as $y = \left(\sin x\right)^4$

$$\therefore \frac{dy}{dx} = 4\left(\sin x\right)^3\cdot\frac{d}{dx}(\sin x) = 4\sin^3 x\cos x$$

Question 13

For this question we will apply each of the differentiation rules informally

(a) By the Product Rule

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^x\right)\cdot\ln x + e^x\cdot\frac{d}{dx}\left(\ln x\right) = \ln x\cdot e^x + e^x\cdot\frac{1}{x} = e^x\left(\ln x + \frac{1}{x}\right)$$

(b) By the Chain Rule

$$\frac{dy}{dx} = -\sin\left(x^4\right)\cdot\frac{d}{dx}\left(x^4\right) = -4x^3\sin\left(x^4\right)$$

(c) As $\cos^4 x = (\cos x)^4$, application of the Composite Rule gives

$$\frac{dy}{dx} = 4\left(\cos x\right)^3\cdot\frac{d}{dx}(\cos x) = -4\sin x\cos^3 x$$

(d) By the Product Rule

$$\frac{dy}{dx} = e^{-x}\cdot\frac{d}{dx}(x) + x\cdot\frac{d}{dx}\left(e^{-x}\right) = e^{-x} - xe^{-x} = e^{-x}(1-x)$$

Question 14

(a) $y = \ln\left(2x^2-1\right)^5 = 5\ln\left(2x^2-1\right)$

Hence, by the Composite and Constant Multiple Rules,

$$\frac{dy}{dx} = 5\times\frac{1}{2x^2-1}\cdot\frac{d}{dx}\left(2x^2-1\right) = \frac{20x}{2x^2-1}$$

(b) $y=\ln\sqrt{\dfrac{x+1}{x-1}}=\ln\left(\dfrac{x+1}{x-1}\right)^{1/2}=\dfrac{1}{2}\ln\left(\dfrac{x+1}{x-1}\right)=\dfrac{1}{2}[\ln(x+1)-\ln(x-1)]$

Hence, by the Composite and Constant Multiple Rules,

$$\frac{dy}{dx}=\frac{1}{2}\left[\frac{1}{x+1}\cdot\frac{d}{dx}(x+1)-\frac{1}{x-1}\cdot\frac{d}{dx}(x-1)\right]=\frac{1}{2}\left(\frac{1}{x+1}-\frac{1}{x-1}\right)=-\frac{1}{(x+1)(x-1)}$$

$$=\frac{1}{1-x^2}$$

(c) $y=\ln\left\{\left(x^2-1\right)^7\left(x^2+1\right)^8\right\}=\ln\left(x^2-1\right)^7+\ln\left(x^2+1\right)^8=7\ln\left(x^2-1\right)+8\ln\left(x^2+1\right)$

Hence, by the Composite and Constant Multiple Rules,

$$\frac{dy}{dx}=7\times\frac{1}{x^2-1}\cdot\frac{d}{dx}\left(x^2-1\right)+8\times\frac{1}{x^2+1}\cdot\frac{d}{dx}\left(x^2+1\right)=\frac{14x}{x^2-1}+\frac{16x}{x^2+1}$$

$$=\frac{14x\left(x^2+1\right)+16x\left(x^2-1\right)}{\left(x^2-1\right)\left(x^2+1\right)}=\frac{30x^3-2x}{\left(x^2-1\right)\left(x^2+1\right)}=\frac{2x\left(15x^2-1\right)}{(x+1)(x-1)\left(x^2+1\right)}$$

Question 15

(a) As $\sec x=\dfrac{1}{\cos x}=(\cos x)^{-1}$, application of the Composite Rule gives

$$\frac{d}{dx}(\sec x)=-(\cos x)^{-2}\frac{d}{dx}(\cos x)=-\frac{1}{\cos^2 x}\cdot(-\sin x)=\frac{1}{\cos x}\cdot\frac{\sin x}{\cos x}=\sec x\tan x$$

(b) Applying the quotient rule (informally) to $\cot x=\dfrac{\cos x}{\sin x}$ gives

$$\frac{d}{dx}(\cot x)=\frac{\sin x(-\sin x)-\cos x\cos x}{\sin^2 x}=-\frac{\sin^2 x+\cos^2 x}{\sin^2 x}=-\frac{1}{\sin^2 x}=-\operatorname{cosec}^2 x$$

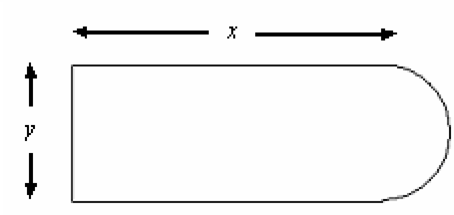
(c) As $\operatorname{cosec} x=\dfrac{1}{\sin x}=(\sin x)^{-1}$, application of the Composite Rule gives

$$\frac{d}{dx}(\operatorname{cosec} x)=-(\sin x)^{-2}\frac{d}{dx}(\sin x)=-\frac{1}{\sin^2 x}\cdot(\cos x)=-\frac{1}{\sin x}\cdot\frac{\cos x}{\sin x}$$

$$=-\operatorname{cosec} x\cot x$$

Question 16

Let the length and width of the sheep pen be x and y metres respectively. Then, using the fact that the circumference and area of a circle are $p\,D$ and $p\,r^2$ respectively, we have



$$\text{Perimeter} = 2x + y + \tfrac{1}{2}p\,y \text{ and } \text{Area} = xy + \tfrac{1}{2}p\left(\tfrac{y}{2}\right)^2 = xy + \tfrac{1}{8}p\,y^2$$

Hence, as the perimeter must be 2000, $2x + y + \tfrac{1}{2}p\,y = 2000 \Rightarrow x = \dfrac{2000 - \tfrac{1}{2}p\,y - y}{2}$

Simplifying this last expression gives $x = \dfrac{1}{4}\left(4000 - p\,y - 2y\right)$ (1)

Substituting this expression for x into the area equation gives

$$A = \frac{1}{4}\left(4000 - p\,y - 2y\right)y + \frac{1}{8}p\,y^2 = 1000y - \frac{1}{8}p\,y^2 - \frac{1}{2}y^2$$

Now the maximum value of A occurs when $\dfrac{dA}{dy} = 0$, where $\dfrac{dA}{dy} = 1000 - \dfrac{1}{4}p\,y - y$.

$$\therefore \frac{dA}{dy} = 0 \Rightarrow 1000 - \frac{1}{4}p\,y - y = 0$$

Multiplying through by 4 gives $4000 - p\,y - 4y = 0$, from which $y = \dfrac{4000}{p + 4}$.

When $y = \dfrac{4000}{p + 4}$, substitution in (1) gives

$$x = \frac{1}{4}\left(4000 - p \cdot \frac{4000}{p + 4} - 2 \cdot \frac{4000}{p + 4}\right) = \frac{1}{4}\left(\frac{4000(p + 4) - 4000p - 8000}{p + 4}\right) = \frac{2000}{p + 4}$$

It remains to show that the area is a maximum (rather than a minimum) when x and y have these values. We apply the Second Derivative Test

As $\dfrac{dA}{dy} = 1000 - \dfrac{1}{4}p\,y - y$, we have $\dfrac{d^2A}{dy^2} = -\dfrac{1}{4}p - 1 < 0$

Hence the area is a maximum when $x = \dfrac{2000}{p + 4}$ and $y = \dfrac{4000}{p + 4}$.