MST 121: Resource Material for Chapter B3, Modelling with vectors

1. Let $\mathbf{a} = -6\mathbf{i} + 8\mathbf{j}$, $\mathbf{b} = -3\mathbf{i} - 11\mathbf{j}$ and $\mathbf{c} = 6\mathbf{i} + \mathbf{j}$

   (a) Write down the column form of each of these vectors.

   (b) Write the vector $\mathbf{d} = 5\mathbf{a} - 2\mathbf{b} + 3\mathbf{c}$ in component form.

   (c) Hence determine the magnitude of $\mathbf{d}$, giving your answer to three significant figures.

   (d) Give the direction in which $\mathbf{d}$ is pointing (i) relative to the direction of the positive x-axis and (ii) as a three figure bearing.

2. (a) A vector $\mathbf{a}$ has magnitude 6 and is directed at 210° to the positive x-axis. Determine the exact component representation of this vector.

   (b) Find the magnitude and direction of each of the following vectors:

   (i) $\mathbf{b} = \sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j}$ (ii) $\mathbf{c} = -\frac{3\sqrt{3}}{2}\mathbf{i} - \frac{3}{2}\mathbf{j}$ (iii) $\mathbf{d} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}$ (iv) $\mathbf{e} = \frac{\sqrt{3}}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}$

3. A coastguard patrolling the waters off Lizard Point travels 8 kilometres due south. He then turns due west and travels for a further 5 kilometres before turning through 60° clockwise to intercept a fishing vessel. Given that he travelled 4 kilometres on the last stage of the journey

   (a) Express each of the three stages of the journey in vector component form

   (b) Hence write down a vector $\mathbf{d}$ that describes the displacement from the starting position to the fishing vessel.

   (c) Use your answer to (b) to write down the distance and the bearing of the fishing vessel from the coastguard's initial position.

   (d) Given that the coastguard travelled at an average speed of 25 kmh^{-1}, how much time would he have saved had he made the journey directly?

   (e) What modelling assumptions have you made in answering (d)?

4. A non-tidal stretch of the River Thames near Henley flows due East at a constant speed of 1.8 ms^{-1}. A girl in a rowing boat, who can row at 2.5 ms^{-1} in still water, starts from a point on the South bank and rows in a direction perpendicular to the bank. Whilst she is crossing the river she is blown off course by a wind with speed 1.4 ms^{-1} from a direction N15°W.

   (a) Find the resultant velocity of the boat, giving both the speed and the bearing correct to three significant figures.

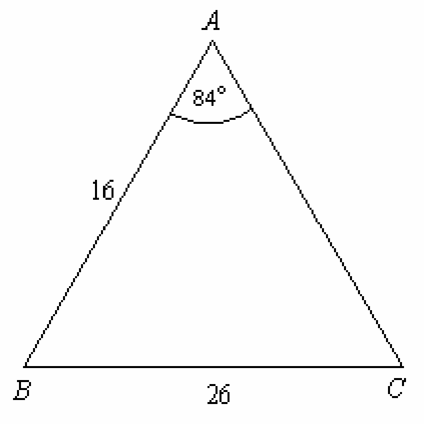
   (b) This particular stretch of river has a constant width of 25 metres. How long does it take the girl to cross the river, and how far upstream or downstream has she then travelled?

5. (a) An aeroplane has a speed of 250 ms^{-1} in still air and is pointed in the direction S30°W. Given that the wind speed is 18 ms^{-1}, blowing from S60°E, use component form to find the velocity of the aeroplane relative to the ground.

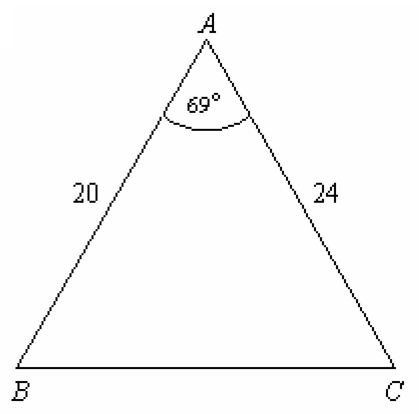
   (b) Hence determine the resultant speed and direction of travel of the aeroplane and use a triangle of velocities to check that your answer is correct.

6. Find all the missing sides and angles in each of the following triangles

(a)



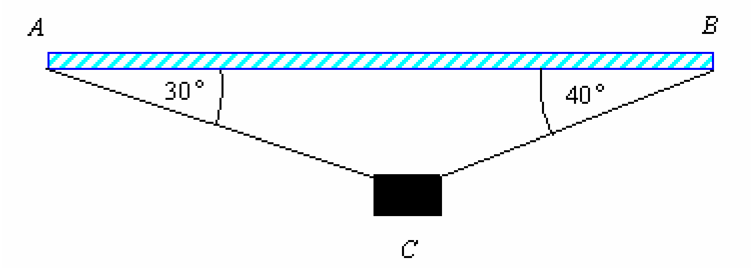
(b)



7. An object which remains at rest is acted on by the forces $\mathbf{F}_1 = 3\mathbf{i} - \mathbf{j}$ N, $\mathbf{F}_2 = 2\mathbf{i} + 5\mathbf{j}$ N and an unknown force $\mathbf{F}_3$.

Use components to find the Cartesian form of the vector $\mathbf{F}_3$. Hence find its magnitude and direction, giving your answers correct to 3 significant figures.

8.

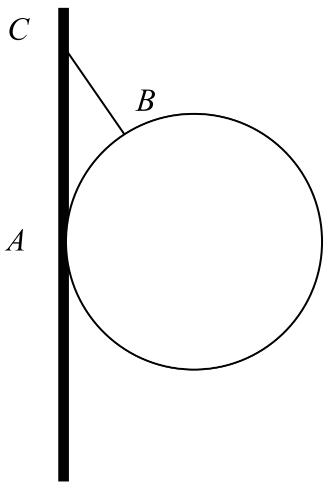


Two strings are attached to the ends *A* and *B* of a horizontal beam, and support a package of mass 5 kg that is attached to the string at the point *C*. When the package hangs in equilibrium $B\hat{A}C = 30^\circ$ and $A\hat{B}C = 40^\circ$. Taking *g* as 10ms^{-2}

(a) Use components to find the tensions in the strings *AC* and *BC*.

(b) Check your answers to (a) by using a triangle of forces.

9.



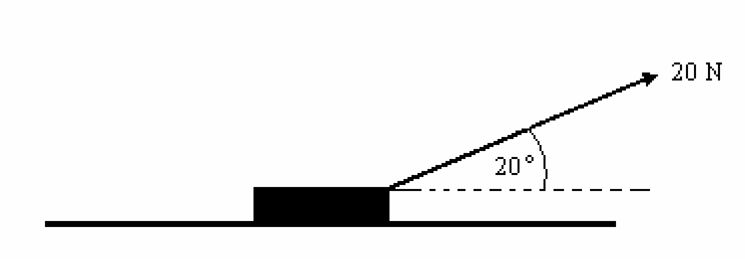
A point *A* on a sphere of radius *r* and weight *W* rests in contact with a smooth vertical wall and is supported by a string of length *r* joining a point *B* on the sphere to a point *C* on the wall.

(a) Draw a diagram showing all the forces acting on the sphere, including the reaction at the wall.

(b) Use components to show that the tension, *T*, in the string is given by $T = \dfrac{2W}{\sqrt{3}}$.

(c) What is the magnitude of the reaction acting at the wall?

10.



A child is attempting to pull a wooden block of mass 2.8 kilograms across the ground by a piece of string inclined at 20° to the horizontal. Given that the block is on the point of moving (limiting equilibrium) when the tension in the string reaches 20 Newtons

(a) What name is given to the resistive force that has so far prevented the block from sliding?

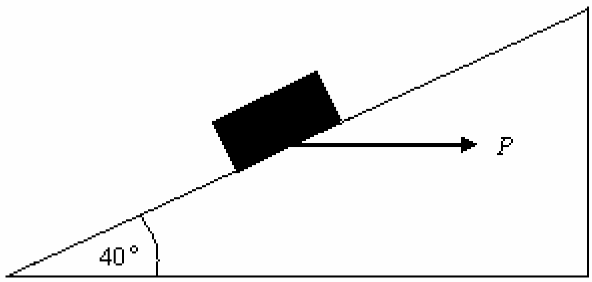
(b) Draw a diagram showing all the forces acting on the block, including the resistive force that you identified in (a).

(c) Determine the magnitude of this resistive force, taking g as 10ms^{-2} and giving your answer correct to three significant figures.

(d) Find the magnitude of the force exerted on the block by the ground.

(e) What is the ratio of magnitude of the resistive force that you calculated in (c) to the magnitude of the normal reaction force that you calculated in (d).

11. A block of mass 2 kg is held at rest on a smooth plane inclined at 40° to the horizontal by a horizontal force P. Taking g as 10ms^{-2}

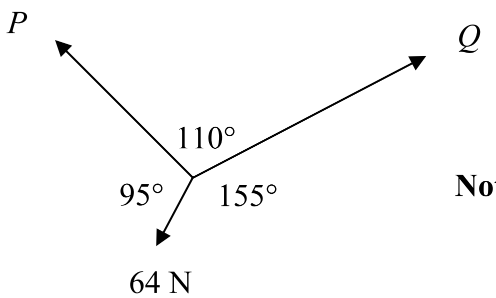


(a) Draw a clearly labelled diagram showing all the forces acting on the particle.

(b) By using components, determine (i) the magnitude of the normal reaction of the plane on the particle and (ii) the magnitude of the force P.

(c) Confirm that your answers to (b) are correct by using a triangle of forces.

12. The following three forces are in equilibrium. Calculate the values of P and Q.



Not to scale

Answers:

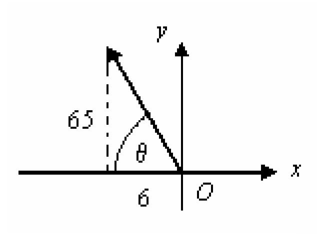
1. (a) $\mathbf{a}=\begin{pmatrix}-6\\8\end{pmatrix}$, $\mathbf{b}=\begin{pmatrix}-3\\-11\end{pmatrix}$ and $\mathbf{c}=\begin{pmatrix}6\\1\end{pmatrix}$

(b) $\mathbf{d}=5\mathbf{a}-2\mathbf{b}+3\mathbf{c}=5\begin{pmatrix}-6\\8\end{pmatrix}-2\begin{pmatrix}-3\\-11\end{pmatrix}+3\begin{pmatrix}6\\1\end{pmatrix}=\begin{pmatrix}-6\\65\end{pmatrix}=-6\mathbf{i}+65\mathbf{j}$

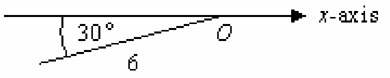
(c) $|\mathbf{d}|=\sqrt{6^2+65^2}=\sqrt{4261}\approx 65.3$

(d) (i) The direction of $\mathbf{d}$ relative to the direction of the positive x-axis is $180-\tan^{-1}\left(\frac{65}{6}\right)\approx 95.3^\circ$

(ii) As a bearing this is $270+\tan^{-1}\left(\frac{65}{6}\right)\approx 355^\circ$

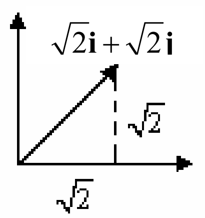


2. (a) $\mathbf{a}=-6\cos 30\mathbf{i}-6\sin 30\mathbf{j}=-3\sqrt{3}\mathbf{i}-3\mathbf{j}$



(b) (i) $|\mathbf{b}|=\sqrt{\left(\sqrt{2}\right)^2+\left(\sqrt{2}\right)^2}=2$

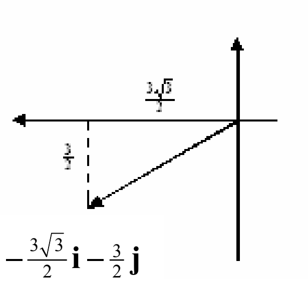
Direction $=\tan^{-1}\left(\dfrac{\sqrt{2}}{\sqrt{2}}\right)=45^\circ$ to the positive x-axis



(ii) $|\mathbf{c}|=\sqrt{\left(\dfrac{3\sqrt{3}}{2}\right)^2+\left(\dfrac{3}{2}\right)^2}=3$

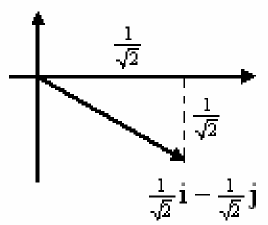
Direction $=180+\tan^{-1}\left(\dfrac{\frac{3}{2}}{\frac{3\sqrt{3}}{2}}\right)$

$=210^\circ$ to the positive x-axis



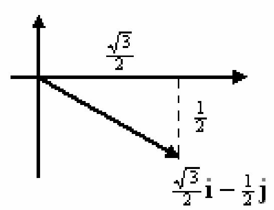
(iii) $|\mathbf{d}|=\sqrt{\left(\frac{1}{\sqrt{2}}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2}=1$

Direction $=-\tan^{-1}\left(\dfrac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}\right)=-45^\circ$



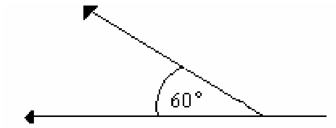
(iv) $|\mathbf{e}|=\sqrt{\left(\frac{\sqrt{3}}{2}\right)^2+\left(\frac{1}{2}\right)^2}=1$

Direction $=-\tan^{-1}\left(\dfrac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)=-30^\circ$



3. (a) The vector representing 8 km due south is $-8\mathbf{j}$

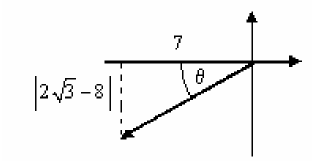
The vector representing 5 km due west is $-5\mathbf{i}$



The vector representing the final stage of the journey is $-4\cos 60\mathbf{i}+4\sin 60\mathbf{j}=-2\mathbf{i}+2\sqrt{3}\mathbf{j}$

(b) $\mathbf{d}=-8\mathbf{j}-5\mathbf{i}+\left(-2\mathbf{i}+2\sqrt{3}\mathbf{j}\right)=-7\mathbf{i}+\left(2\sqrt{3}-8\right)\mathbf{j}$

(c) $\text{Distance}=\left|\mathbf{d}\right|=\sqrt{7^2+\left(2\sqrt{3}-8\right)^2}\approx 8.34\text{ km}$



$$\text{Bearing}=270-\theta=270-\tan^{-1}\left(\frac{\left|2\sqrt{3}-8\right|}{7}\right)\approx 237^\circ$$

(d)

$$\text{Time of actual journey}=\frac{8+5+4}{25}=0.68\text{ hrs }\approx 41\text{ mins}$$

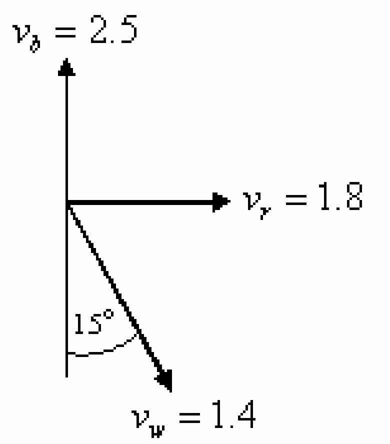
$$\text{Time of direct journey}=\frac{8.34}{25}\approx 0.336\text{ hrs }\approx 20\text{ mins}$$

Hence he would have saved approximately 21 minutes.

(e) We have implicitly assumed that the affects of the wind and tide would not have prevented him from maintaining the same constant speed of 25 kmh^{-1} on both journeys.

4. (a) If the resultant velocity of the boat is given by $\mathbf{v}$, we have

$$\begin{aligned}\mathbf{v}&=\mathbf{v}_b+\mathbf{v}_r+\mathbf{v}_w\\&=2.5\mathbf{j}+1.8\mathbf{i}+1.4\left(\sin 15\mathbf{i}-\cos 15\mathbf{j}\right)\\&=\left(1.8+1.4\sin 15\right)\mathbf{i}+\left(2.5-1.4\cos 15\right)\mathbf{j}\\&\approx 2.162\mathbf{i}+1.148\mathbf{j}\end{aligned}$$



Hence the magnitude of the resultant velocity is $\left|\mathbf{v}\right|=\sqrt{2.162^2+1.148^2}\approx 2.45\text{ ms}^{-1}$.

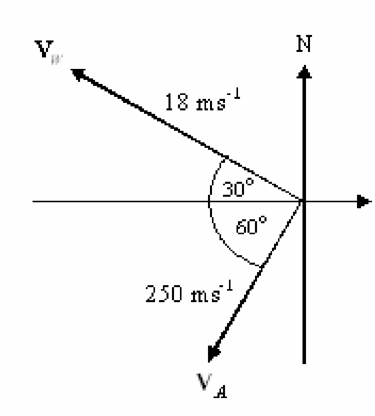
And its direction (as a bearing from the north line) is $90-\tan^{-1}\left(\dfrac{1.148}{2.162}\right)\approx 062^\circ$.

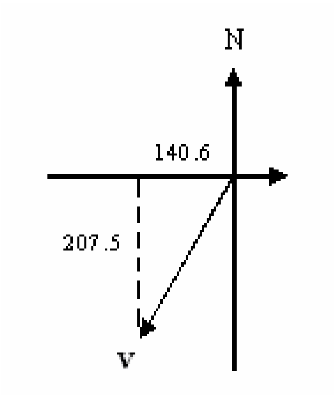
(b) As the river is 25 metres wide, it takes $\frac{25}{1.148}\approx 21.8$ seconds to cross it. During this time the boat will have travelled downstream through a total distance of $21.8\times 2.162\approx 47.1$ metres.

5. (a) If the resultant velocity of the aircraft is $\mathbf{v}$, then

$$\begin{aligned}\mathbf{v} &= \mathbf{v}_A + \mathbf{v}_W \\ &= (-250\cos 60\mathbf{i} - 250\sin 60\mathbf{j}) + (-18\cos 30\mathbf{i} + 18\sin 30\mathbf{j}) \\ &= -(250\cos 60 + 18\cos 30)\mathbf{i} + (18\sin 30 - 250\sin 60)\mathbf{j} \\ &\approx -140.6\mathbf{i} - 207.5\mathbf{j}\end{aligned}$$

Not to scale

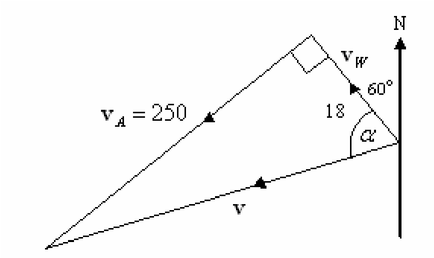




(b) $\therefore |\mathbf{v}| = \sqrt{140.6^2 + 207.5^2} \approx 250.6 \text{ ms}^{-1}$

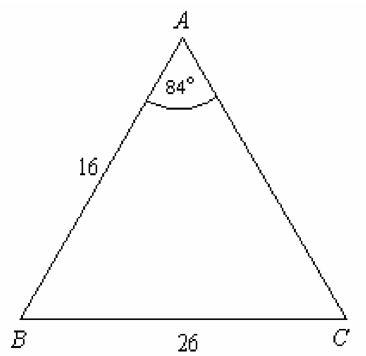
As a bearing, its direction is given by $270 - \tan^{-1}\left(\dfrac{207.5}{140.6}\right) \approx 214^\circ$

$$\begin{aligned}|\mathbf{v}| &= \sqrt{|v_A|^2 + |v_B|^2} \\ &= \sqrt{250^2 + 18^2} \\ &\approx 250.6 \text{ ms}^{-1}\end{aligned}$$



The velocity is directed on a bearing of $360 - 60 - \alpha = 300 - \tan^{-1}\left(\frac{250}{18}\right) \approx 214^\circ$

6. (a)



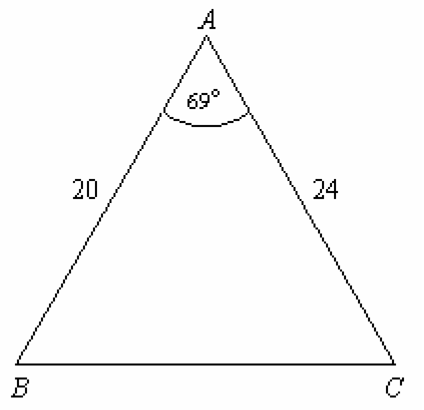
By the sine rule $\dfrac{\sin C}{16} = \dfrac{\sin 84}{26}$ so $C \approx 37.8^\circ$

[Note that C cannot be $180 - 37.8 = 142.3^\circ$, for the angle sum of the triangle would then exceed 180°.]

$\therefore \hat{B} \approx 58.3^\circ$ (As angle sum of $\Delta = 180^\circ$)

By the cosine rule, $AC = \sqrt{16^2 + 26^2 - 2\times 16\times 26\times \cos 58.3} \approx 22.2$

(b)



By the cosine rule

$$BC = \sqrt{20^2 + 24^2 - 2\times 20\times 24\times \cos 69}$$
$$\approx 25.1$$

$\therefore$ By the sine rule

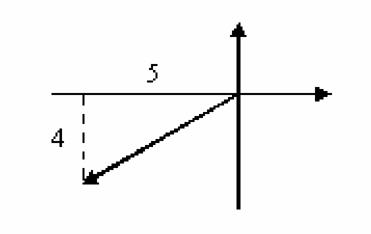
$$\frac{\sin B}{24} = \frac{\sin 69}{25.1} \Rightarrow B \approx 63.0^\circ$$

[Note that C cannot be $180-63=117^\circ$, for the angle sum of the triangle would then exceed 180°.]

$\therefore C \approx 48.0^\circ$ (as angle sum of $\Delta = 180^\circ$).

7. As $\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = \mathbf{0}$ then $\mathbf{F}_3 = -\mathbf{F}_1 - \mathbf{F}_2 = -(3\mathbf{i}-\mathbf{j})-(2\mathbf{i}+5\mathbf{j}) = -5\mathbf{i}-4\mathbf{j}$

$\therefore |\mathbf{F}_3| = \sqrt{5^2+4^2} \approx 6.40$ N



The direction of $\mathbf{F}_3$ makes an angle of $180 - \tan^{-1}\left(\frac{4}{5}\right) \approx 141^\circ$ with the direction of the positive x-axis (or -141°).

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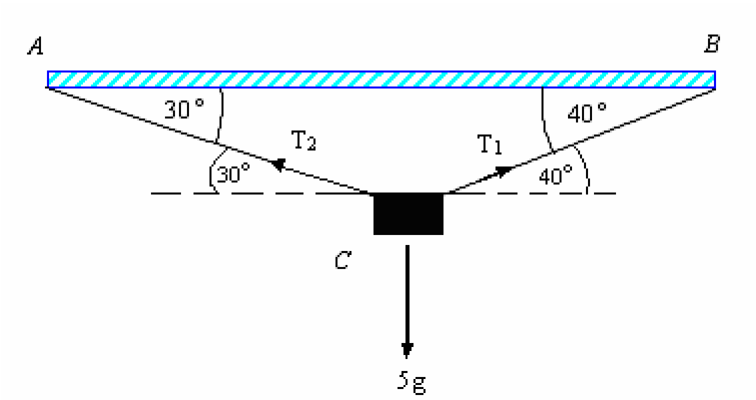


(a) The vector sum of the forces is given by

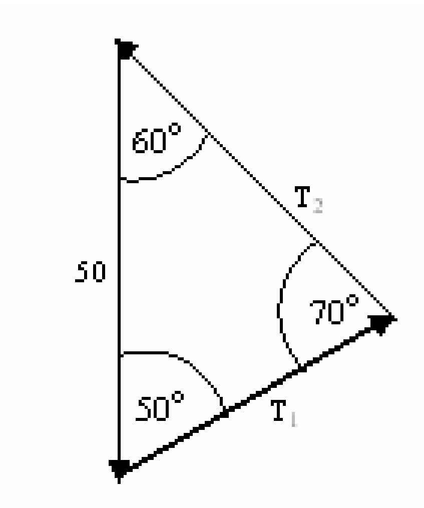
$$\mathbf{F} = \left(T_1\cos 40\mathbf{i} + T_1 \sin 40\mathbf{j}\right) + \left(-T_2\cos 30\mathbf{i} + T_2\sin 30\mathbf{j}\right) - 5g\mathbf{j}$$
$$= \left(T_1\cos 40 - T_2\cos 30\right)\mathbf{i} + \left(T_1\sin 40 + \tfrac{1}{2}T_2 - 50\right)\mathbf{j}$$

Hence, since $\mathbf{F} = \mathbf{0}$ (as the package is in equilibrium) we have

$T_1\cos 40 - T_2\cos 30 = 0$ and $T_1\sin 40 + \frac{1}{2}T_2 - 50 = 0$

Solving these equations gives $T_1 \approx 46.1$ N and $T_2 \approx 40.8$ N.

(b) Constructing a triangle of forces to model the problem gives



Applying the sine rule gives

$$\frac{T_1}{\sin 60} = \frac{50}{\sin 70} \Rightarrow T_1 = \frac{50\sin 60}{\sin 70} \approx 46.1 \text{ N}$$

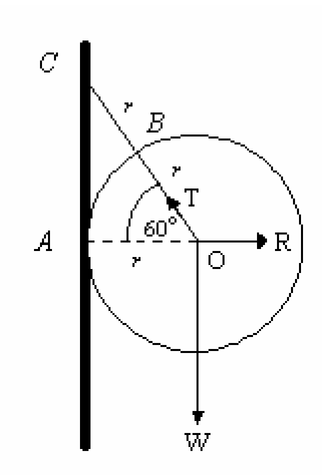
$$\frac{T_2}{\sin 50} = \frac{50}{\sin 70} \Rightarrow T_2 = \frac{50\sin 50}{\sin 70} \approx 40.8 \text{ N}$$

which are in agreement with the results that we obtained by using the component forms.

9. It is generally true that if three non-parallel forces are in equilibrium, the lines of action of these forces meet at a point. In this case, the point where the three lines of action meet is the centre of the sphere.

(a)

(b)



$A\hat{O}C = \cos^{-1}\left(\frac{r}{2r}\right) = 60^\circ$ (as $OA = r$ and $OC = 2r$)

The vector sum of all the forces is given by

$$\begin{aligned}\mathbf{F} &= R\mathbf{i} - W\mathbf{j} - T\cos 60\mathbf{i} + T\sin 60\mathbf{j} \\ &= (R - T\cos 60)\mathbf{i} + (T\sin 60 - W)\mathbf{j}\end{aligned}$$

As the sphere is in equilibrium we know that $\mathbf{F} = \mathbf{0}$. Hence the $\mathbf{j}$ component is 0 and we have

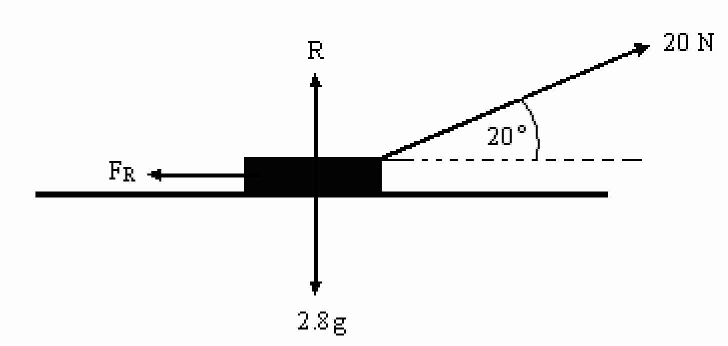
$$T\sin 60 - W = 0 \Rightarrow T\cdot\frac{\sqrt{3}}{2} - W = 0 \text{ and } T = \frac{2W}{\sqrt{3}}$$

(c) As the $\mathbf{i}$ component of $\mathbf{F}$ is also 0, we know that $R - T\cos 60 = 0$.

Hence $R = T\cos 60 = \frac{2W}{\sqrt{3}}\cdot\frac{1}{2} = \frac{W}{\sqrt{3}}$.

10. (a) Friction

(b)

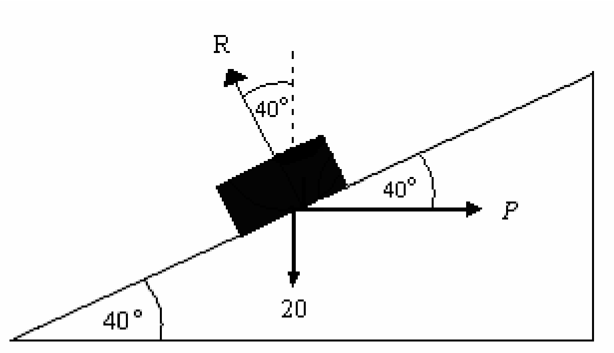


(c) Taking horizontal components gives the nett horizontal force acting on the block as $(20\cos 20 - F_R)\mathbf{i}$. But as the block is in equilibrium, we know that this is $\mathbf{0}$. Hence $F_R = 20\cos 20 \approx 18.8$ N.

(d) Taking vertical components gives the nett vertical force acting on the block as $(R + 20\sin 20 - 28)\mathbf{j}$. As this is also $\mathbf{0}$, $R = 28 - 20\sin 20 \approx 21.2$ N.

(e) $\dfrac{\text{magnitude of frictional force}}{\text{magnitude of normal reaction}} = \dfrac{18.8}{21.2} \approx 0.89$

11. (a)



(b) The total force acting on the block is given by

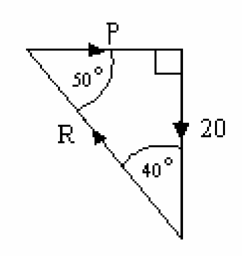
$$F = P\mathbf{i} - 20\mathbf{j} + (-R\sin 40\mathbf{i} + R\cos 40\mathbf{j}) = (P - R\sin 40)\mathbf{j} + (R\cos 40 - 20)\mathbf{j}$$

As the block is in equilibrium, we know that $\mathbf{F} = \mathbf{0}$. Hence each of the components must also be 0 and we have

(i) $R\cos 40 - 20 = 0 \Rightarrow R \approx 26.1$ N

(ii) $P = R\sin 40 \approx 16.8$ N

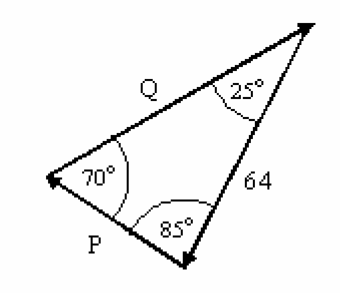
(c)



$P = 20\tan 40 \approx 16.8$ N

$$R = \frac{20}{\cos 40} \approx 26.1 \text{ N}$$

12.



By the sine rule

$$\frac{Q}{\sin 85} = \frac{64}{\sin 70} \Rightarrow Q \approx 67.8 \text{ N}$$

$$\frac{P}{\sin 25} = \frac{64}{\sin 70} \Rightarrow P \approx 28.8 \text{ N}$$