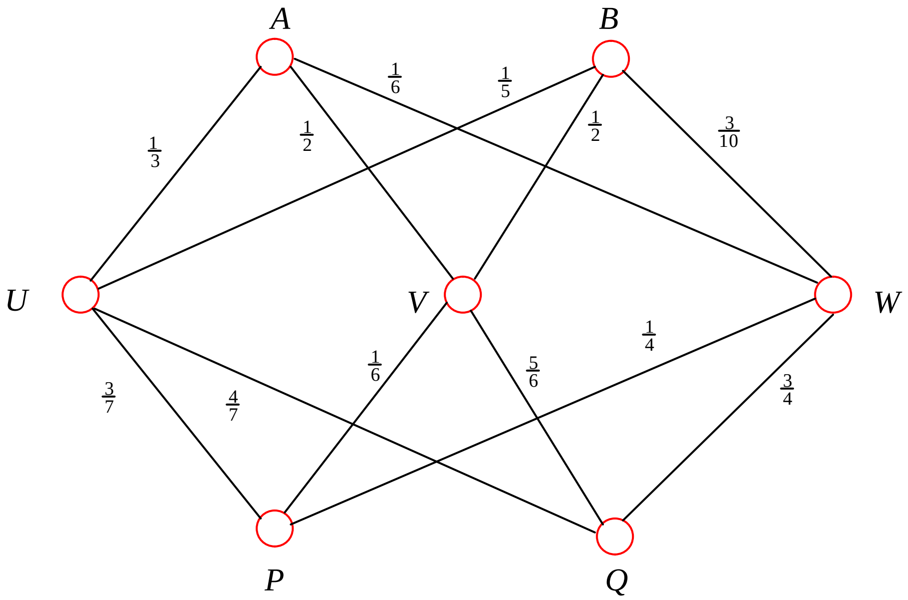
MST 121: Resource Material for Chapter B2, Modelling with matrices

1. This question is concerned with modelling the flow in the network below



(a) Write down the matrix that models the flow in the network from the nodes at A and B to the nodes at U, V and W.

(b) Write down the matrix that models the flow in the network from the nodes at U, V and W to the nodes at P and Q.

(c) Hence write down a single matrix that models the flow in the network from the nodes at A and B to the nodes at P and Q.

(d) If the input at A and B is a and b respectively, use the matrix that you calculated in (c) to determine the output p and q at P and Q.

2. Let $\mathbf{A} = \begin{pmatrix} 3 & -2 \\ 4 & 7 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 5 & 0 & 9 \\ -3 & 7 & 8 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 2 & 5 \\ -3 & -1 \\ 6 & 2 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} 4 & 1 & -9 \\ 2 & 0 & -1 \\ 0 & 3 & 7 \end{pmatrix}$

For each of the following product matrices, either form the matrix product or explain why it cannot be formed:

(a) **AB** (b) **BA** (c) **BC** (d) **DC** (e) **CA** (f) **BD** (g) **AD** (h) **AC**

3. Let $\mathbf{A} = \begin{pmatrix} 4 & -5 \\ 6 & 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 9 & -5 \\ 7 & 8 \end{pmatrix}$. Form the matrix products **AB** and **BA** and hence prove that matrix multiplication is not commutative.

4. Let $\mathbf{A} = \begin{pmatrix} 2 & 7 \\ -6 & 8 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 4 & 9 \\ -8 & 3 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} -5 & 4 \\ 2 & 8 \end{pmatrix}$. Form the matrix products $\mathbf{A}(\mathbf{BC})$ and $(\mathbf{AB})\mathbf{C}$ and hence demonstrate that this particular choice of matrices satisfies the associativity property of matrix multiplication.

5. Explain why your answer to question 3 constituted a proof that matrix multiplication is not commutative, but why your answer to question 4 did not prove that matrix multiplication is associative.

6. Let $\mathbf{A}$ be the matrix $\begin{pmatrix} -2 & 7 \\ -3 & 8 \end{pmatrix}$. Determine the matrix powers $\mathbf{A}^2$, $\mathbf{A}^3$ and $\mathbf{A}^4$.

7. Let $\mathbf{A}=\begin{pmatrix} 5 & 2 \\ -9 & -7 \end{pmatrix}$, $\mathbf{B}=\begin{pmatrix} 4 & 6 \\ 0 & -3 \end{pmatrix}$ and $\mathbf{C}=\begin{pmatrix} 2 & 6 \\ 8 & 5 \end{pmatrix}$.

Determine (a) $3\mathbf{A}-2\mathbf{B}$ (b) $2\mathbf{A}-3\mathbf{B}-4\mathbf{C}$ (c) $\mathbf{A}^2+3\mathbf{BC}$ (d) $\mathbf{A}^3-\mathbf{B}^3+2\mathbf{C}$

8. In Activity 2.7 on page 24 of the chapter, we saw that the expression $\mathbf{AB}+\mathbf{AC}$ was equivalent to the factorised expression $\mathbf{A}(\mathbf{B}+\mathbf{C})$. This is equivalent to saying that matrix multiplication is distributive over addition.

Use the matrices $\mathbf{A}=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $\mathbf{B}=\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ and $\mathbf{C}=\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$ to prove that this result is true for ***all*** 2×2 matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$.

9. At the start of the year 2525, a colony of 1000 humans decides to settle on the planet Vulcan. The birth and death rates for this colony are tabulated below, and you may assume that $\frac{1}{15}$ of the surviving juveniles become adults each year:

	Juveniles (aged<15)	Adults (aged ≥ 15)
Subpopulation	134	866
Birth Rate	0.0000	0.0482
Death Rate	0.026	0.0317

(a) Use a network diagram to help you to construct a transition matrix to model the change in the population from year n to year $n+1$.

(b) Use your transition matrix to determine the numbers of adults and juniors at the start of the years 2526, 2527 and 2528.

(c) If there were 424 juveniles and 817 adults at the start of the year 2601, determine the numbers in each subpopulation at the start of the year 2600.

10. (a) Calculate the determinants of each of the following matrices.

$$\mathbf{A}=\begin{pmatrix} 2 & 4 \\ 4 & 8 \end{pmatrix}, \mathbf{B}=\begin{pmatrix} 3 & 7 \\ -2 & 5 \end{pmatrix}, \mathbf{C}=\begin{pmatrix} -3 & 1 \\ 0 & 6 \end{pmatrix} \text{ and } \mathbf{D}=\begin{pmatrix} 2 & 6 \\ 3 & 9 \end{pmatrix}$$

(b) For each of the matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ either calculate the inverse or give a reason why the inverse does not exist.

11. Use matrix-vector methods to solve each of the following pairs of linear simultaneous equations:

$$(a)\quad \begin{aligned} 3x + 7y &= 61 \\ 4x - 2y &= 2 \end{aligned} \qquad (b)\quad \begin{aligned} 7x - 4y &= -23 \\ 3x + 9y &= -42 \end{aligned} \qquad (c)\quad \begin{aligned} 2x - 11y &= 90 \\ 3x + 17y &= 1 \end{aligned}$$

The remainder of the questions on this worksheet are designed to give you further practice with manipulating matrices. You do ***not*** need to learn or remember any of the definitions that are introduced.

12. (a) If two matrices $\mathbf{A}$ and $\mathbf{B}$ have the property that $\mathbf{AB} = -\mathbf{BA}$, the matrices are said to **anti-commute**.

Use this definition to show that $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 4 & -1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}$ are a pair of anti-commutative matrices.

(b) A matrix $\mathbf{A}$ for which $\mathbf{A}^{k+1} = \mathbf{A}$, where $k \in \mathbb{N}$, is called **periodic**. If k is the smallest such integer for which $\mathbf{A}$ has this property, then $\mathbf{A}$ is said to have **period *k***.

Use this definition to show that $\mathbf{A} = \begin{pmatrix} -4 & 1 \\ -13 & 3 \end{pmatrix}$ is periodic of order 3.

(c) A periodic matrix for which $k = 1$, so that $\mathbf{A}^2 = \mathbf{A}$, is called an **idempotent matrix**. Use this definition to verify that $\mathbf{A} = \begin{pmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{pmatrix}$ is idempotent.

(d) A matrix $\mathbf{A}$ for which $\mathbf{A}^p = \mathbf{0}$, where $p \in \mathbb{N}$, is called a **nilpotent** matrix. If p is the least positive integer for which $\mathbf{A}$ has this property, then $\mathbf{A}$ is said to be **nilpotent of index (**or **order) p**.

Use this definition to show that $\mathbf{A} = \begin{pmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{pmatrix}$ is nilpotent of order 3.

(e) A matrix $\mathbf{A}$ for which $\mathbf{A}^2 = \mathbf{I}$ is called **involutory.** It follows that all involutory matrices are their own inverses (self inverse).

Show that $\mathbf{A} = \begin{pmatrix} 3 & -4 \\ 2 & -3 \end{pmatrix}$ is an involutory matrix, and that it satisfies the following general properties of involutory matrices

- $\frac{1}{2}(\mathbf{I} + \mathbf{A})$ and $\frac{1}{2}(\mathbf{I} - \mathbf{A})$ are idempotent (see (c) above)
- $(\mathbf{I} + \mathbf{A}) \cdot (\mathbf{I} - \mathbf{A}) = \mathbf{0}$, where $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the 2×2 identity matrix.

Answers:

1. (a) $\begin{pmatrix} \frac{1}{3} & \frac{1}{5} \\ \frac{1}{2} & \frac{1}{2} \\ \frac{1}{6} & \frac{3}{10} \end{pmatrix}$ (b) $\begin{pmatrix} \frac{3}{7} & \frac{1}{6} & \frac{1}{4} \\ \frac{4}{7} & \frac{5}{6} & \frac{3}{4} \end{pmatrix}$ (c) $\begin{pmatrix} \frac{3}{7} & \frac{1}{6} & \frac{1}{4} \\ \frac{4}{7} & \frac{5}{6} & \frac{3}{4} \end{pmatrix}\begin{pmatrix} \frac{1}{3} & \frac{1}{5} \\ \frac{1}{2} & \frac{1}{2} \\ \frac{1}{6} & \frac{3}{10} \end{pmatrix} = \begin{pmatrix} \frac{15}{56} & \frac{41}{168} \\ \frac{41}{56} & \frac{127}{168} \end{pmatrix}$

   (d) $\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} \frac{15}{56} & \frac{41}{168} \\ \frac{41}{56} & \frac{127}{168} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \frac{15}{56}a + \frac{41}{168}b \\ \frac{41}{56}a + \frac{127}{168}b \end{pmatrix}$. Hence $p = \frac{15}{56}a + \frac{41}{168}b$ and $q = \frac{41}{56}a + \frac{127}{168}b$.

2. (a) $\mathbf{AB} = \begin{pmatrix} 21 & -14 & 11 \\ -1 & 49 & 92 \end{pmatrix}$

   (b) $\mathbf{BA}$ cannot be formed because the number of columns of $\mathbf{B}$ is not equal to the number of rows of $\mathbf{A}$.

   (c) $\mathbf{BC} = \begin{pmatrix} 64 & 43 \\ 21 & -6 \end{pmatrix}$

   (d) $\mathbf{DC} = \begin{pmatrix} -49 & 1 \\ -2 & 8 \\ 33 & 11 \end{pmatrix}$

   (e) $\mathbf{CA} = \begin{pmatrix} 26 & 31 \\ -13 & -1 \\ 26 & 2 \end{pmatrix}$

   (f) $\mathbf{BD} = \begin{pmatrix} 20 & 32 & 18 \\ 2 & 21 & 76 \end{pmatrix}$

   (g) $\mathbf{AD}$ cannot be formed because the number of columns of $\mathbf{A}$ is not equal to the number of rows of $\mathbf{D}$.

   (h) $\mathbf{AC}$ cannot be formed because the number of columns of $\mathbf{A}$ is not equal to the number of rows of $\mathbf{C}$.

3. $\mathbf{AB} = \begin{pmatrix} 1 & -60 \\ 68 & -14 \end{pmatrix}$ and $\mathbf{BA} = \begin{pmatrix} 6 & -55 \\ 76 & -19 \end{pmatrix}$. Hence $\mathbf{AB} \neq \mathbf{BA}$.

   This provides a single counter example and is sufficient to prove that, in general, matrix multiplication is not commutative.

4. Both $\mathbf{A}(\mathbf{BC})$ and $(\mathbf{AB})\mathbf{C}$ are equal to $\begin{pmatrix} 318 & 120 \\ 380 & -592 \end{pmatrix}$, and hence the matrices $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ satisfy the associative property of matrix multiplication.

5. To prove that a result is not true (question 3), it is sufficient to find a single counter example. However, to show that a result is always true (question 4), you need to construct a proof in general terms that covers all possible cases (see the solution to question 8).

6. $\mathbf{A}^2 = \begin{pmatrix} -17 & 42 \\ -18 & 43 \end{pmatrix}$, $\mathbf{A}^3 = \begin{pmatrix} -92 & 217 \\ -93 & 218 \end{pmatrix}$ and $\mathbf{A}^4 = \begin{pmatrix} -467 & 1092 \\ -468 & 1093 \end{pmatrix}$

7. $(a)\ \begin{pmatrix} 7 & -6 \\ -27 & -15 \end{pmatrix}$ $(b)\ \begin{pmatrix} -10 & -38 \\ -50 & -25 \end{pmatrix}$ $(c)\ \begin{pmatrix} 175 & 158 \\ -54 & -14 \end{pmatrix}$ $(d)\ \begin{pmatrix} 11 & -24 \\ -173 & -144 \end{pmatrix}$

8. With $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$, we have

$$\mathbf{AB} + \mathbf{AC} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix} + \begin{pmatrix} a_{11}c_{11} + a_{12}c_{21} & a_{11}c_{12} + a_{12}c_{22} \\ a_{21}c_{11} + a_{22}c_{21} & a_{21}c_{12} + a_{22}c_{22} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{11}c_{11} + a_{12}c_{21} & a_{11}b_{12} + a_{12}b_{22} + a_{11}c_{12} + a_{12}c_{22} \\ a_{21}b_{11} + a_{22}b_{21} + a_{21}c_{11} + a_{22}c_{21} & a_{21}b_{12} + a_{22}b_{22} + a_{21}c_{12} + a_{22}c_{22} \end{pmatrix}$$

Also,

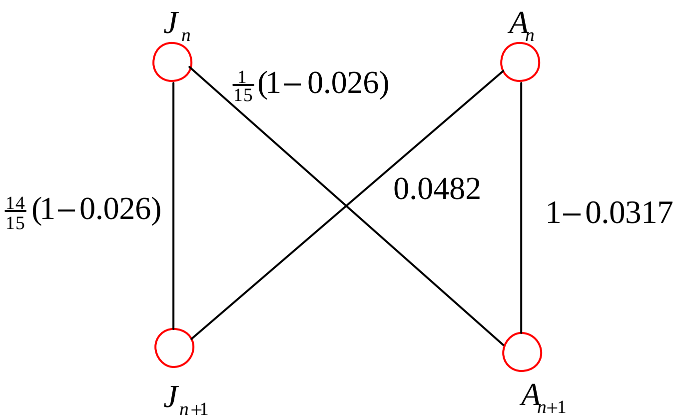
$$\mathbf{A}(\mathbf{B} + \mathbf{C}) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \times \begin{pmatrix} b_{11} + c_{11} & b_{12} + c_{12} \\ b_{21} + c_{21} & b_{22} + c_{22} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}(b_{11} + c_{11}) + a_{12}(b_{21} + c_{21}) & a_{11}(b_{12} + c_{12}) + a_{12}(b_{22} + c_{22}) \\ a_{21}(b_{11} + c_{11}) + a_{22}(b_{21} + c_{21}) & a_{21}(b_{12} + c_{12}) + a_{22}(b_{22} + c_{22}) \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{11}c_{11} + a_{12}c_{21} & a_{11}b_{12} + a_{12}b_{22} + a_{11}c_{12} + a_{12}c_{22} \\ a_{21}b_{11} + a_{22}b_{21} + a_{21}c_{11} + a_{22}c_{21} & a_{21}b_{12} + a_{22}b_{22} + a_{21}c_{12} + a_{22}c_{22} \end{pmatrix}$$

Hence $\mathbf{AB} + \mathbf{AC} = \mathbf{A}(\mathbf{B} + \mathbf{C})$, which completes the proof that, for 2×2 matrices, multiplication is distributive over addition.

9. (a)



$$\therefore \begin{pmatrix} J_{n+1} \\ A_{n+1} \end{pmatrix} = \begin{pmatrix} 0.9091 & 0.0482 \\ 0.0649 & 0.9683 \end{pmatrix} \begin{pmatrix} J_n \\ A_n \end{pmatrix}$$, and so with $\begin{pmatrix} J_0 \\ A_0 \end{pmatrix} = \begin{pmatrix} 134 \\ 866 \end{pmatrix}$ we have

(b) Population at the start of 2526 is $\begin{pmatrix} J_1 \\ A_1 \end{pmatrix} = \begin{pmatrix} 0.9091 & 0.0482 \\ 0.0649 & 0.9683 \end{pmatrix}\begin{pmatrix} 134 \\ 866 \end{pmatrix} = \begin{pmatrix} 164 \\ 847 \end{pmatrix}$

Population at the start of 2527 is $\begin{pmatrix} J_2 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0.9091 & 0.0482 \\ 0.0649 & 0.9683 \end{pmatrix}\begin{pmatrix} 164 \\ 847 \end{pmatrix} = \begin{pmatrix} 190 \\ 831 \end{pmatrix}$

Population at the start of 2528 is $\begin{pmatrix} J_3 \\ A_3 \end{pmatrix} = \begin{pmatrix} 0.9091 & 0.0482 \\ 0.0649 & 0.9683 \end{pmatrix}\begin{pmatrix} 190 \\ 831 \end{pmatrix} = \begin{pmatrix} 212 \\ 817 \end{pmatrix}$

(c) If there were 424 juveniles and 817 adults at the start of 2601, we know that $\begin{pmatrix} 0.9091 & 0.0482 \\ 0.0649 & 0.9683 \end{pmatrix}\begin{pmatrix} J_{74} \\ A_{74} \end{pmatrix} = \begin{pmatrix} 424 \\ 817 \end{pmatrix}$. In order to undo the effects of this matrix we need to multiply it by its inverse. Hence

$$\begin{pmatrix} J_{74} \\ A_{74} \end{pmatrix} = \begin{pmatrix} 0.9091 & 0.0482 \\ 0.0649 & 0.9683 \end{pmatrix}^{-1}\begin{pmatrix} 424 \\ 817 \end{pmatrix} \approx \begin{pmatrix} 1.1039 & -0.0550 \\ -0.0740 & 1.0364 \end{pmatrix}\begin{pmatrix} 424 \\ 817 \end{pmatrix} = \begin{pmatrix} 423 \\ 815 \end{pmatrix}$$

So there would have been 423 juveniles and 815 adults at the start of 2600.

10. (a) $\det \mathbf{A} = 0$, $\det \mathbf{B} = 29$, $\det \mathbf{C} = -18$ and $\det \mathbf{D} = 0$

(b) $\mathbf{A}$ does not have an inverse because its determinant is zero

$$\mathbf{B}^{-1} = \tfrac{1}{29}\begin{pmatrix} 5 & -7 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} \frac{5}{29} & -\frac{7}{29} \\ \frac{2}{29} & \frac{3}{29} \end{pmatrix}$$

$$\mathbf{C}^{-1} = -\tfrac{1}{18}\begin{pmatrix} 6 & -1 \\ 0 & -3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} & \frac{1}{18} \\ 0 & \frac{1}{6} \end{pmatrix}$$

$\mathbf{D}$ does not have an inverse because its determinant is zero

11. (a) We need to solve $\begin{pmatrix} 3 & 7 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 61 \\ 2 \end{pmatrix}$

$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & 7 \\ 4 & -2 \end{pmatrix}^{-1}\begin{pmatrix} 61 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{17} & \frac{7}{34} \\ \frac{2}{17} & -\frac{3}{34} \end{pmatrix}\begin{pmatrix} 61 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$. Hence $x = 4$ and $y = 7$

(b) We need to solve $\begin{pmatrix} 7 & -4 \\ 3 & 9 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -23 \\ -42 \end{pmatrix}$

$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 & -4 \\ 3 & 9 \end{pmatrix}^{-1}\begin{pmatrix} -23 \\ -42 \end{pmatrix} = \begin{pmatrix} \frac{3}{25} & \frac{4}{75} \\ -\frac{1}{25} & \frac{7}{75} \end{pmatrix}\begin{pmatrix} -23 \\ -42 \end{pmatrix} = \begin{pmatrix} -5 \\ -3 \end{pmatrix}$. Hence $x = -5$ and $y = -3$

(c) We need to solve $\begin{pmatrix} 2 & -11 \\ 3 & 17 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 90 \\ 1 \end{pmatrix}$

$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -11 \\ 3 & 17 \end{pmatrix}^{-1}\begin{pmatrix} 90 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{17}{67} & \frac{11}{67} \\ -\frac{3}{67} & \frac{2}{67} \end{pmatrix}\begin{pmatrix} 90 \\ 1 \end{pmatrix} = \begin{pmatrix} 23 \\ -4 \end{pmatrix}$. Hence $x = 23$ and $y = -4$

12. (a) We have $\mathbf{AB}=\begin{pmatrix}3 & -2\\ 2 & -3\end{pmatrix}$ and $\mathbf{BA}=\begin{pmatrix}-3 & 2\\ -2 & 3\end{pmatrix}$.

Hence $\mathbf{AB}=-\mathbf{BA}$ and so $\mathbf{A}$ and $\mathbf{B}$ are a pair of anti-commutative matrices.

(b) We need to show that $\mathbf{A}^4=\mathbf{A}$, but that $\mathbf{A}^2\neq\mathbf{A}$ and $\mathbf{A}^3\neq\mathbf{A}$. We have

$$\mathbf{A}^2=\begin{pmatrix}-4 & 1\\ -13 & 3\end{pmatrix}\begin{pmatrix}-4 & 1\\ -13 & 3\end{pmatrix}=\begin{pmatrix}3 & -1\\ 13 & -4\end{pmatrix}\neq\mathbf{A}$$

$$\mathbf{A}^3=\mathbf{A}^2\cdot\mathbf{A}=\begin{pmatrix}3 & -1\\ 13 & -4\end{pmatrix}\begin{pmatrix}-4 & 1\\ -13 & 3\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\neq\mathbf{A}$$

$$\mathbf{A}^4=\mathbf{A}^3\cdot\mathbf{A}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}-4 & 1\\ -13 & 3\end{pmatrix}=\begin{pmatrix}-4 & 1\\ -13 & 3\end{pmatrix}=\mathbf{A}$$

Hence $\mathbf{A}$ is periodic, with period 3.

(c) $$\mathbf{A}^2=\begin{pmatrix}2 & -3 & -5\\ -1 & 4 & 5\\ 1 & -3 & -4\end{pmatrix}\begin{pmatrix}2 & -3 & -5\\ -1 & 4 & 5\\ 1 & -3 & -4\end{pmatrix}=\begin{pmatrix}2 & -3 & -5\\ -1 & 4 & 5\\ 1 & -3 & -4\end{pmatrix}$$

Hence $\mathbf{A}$ is idempotent.

(d) We need to show that $\mathbf{A}^3=\mathbf{0}$, but that $\mathbf{A}^2\neq\mathbf{0}$. We have

$$\mathbf{A}^2=\begin{pmatrix}1 & 1 & 3\\ 5 & 2 & 6\\ -2 & -1 & -3\end{pmatrix}\begin{pmatrix}1 & 1 & 3\\ 5 & 2 & 6\\ -2 & -1 & -3\end{pmatrix}=\begin{pmatrix}0 & 0 & 0\\ 3 & 3 & 9\\ -1 & -1 & -3\end{pmatrix}$$

$$\mathbf{A}^3=\mathbf{A}^2\cdot\mathbf{A}=\begin{pmatrix}0 & 0 & 0\\ 3 & 3 & 9\\ -1 & -1 & -3\end{pmatrix}\begin{pmatrix}1 & 1 & 3\\ 5 & 2 & 6\\ -2 & -1 & -3\end{pmatrix}=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}=\mathbf{0}$$

Hence $\mathbf{A}$ is nilpotent of order 3.

(e) $A^2=\begin{pmatrix}3 & -4\\ 2 & -3\end{pmatrix}\begin{pmatrix}3 & -4\\ 2 & -3\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=\mathbf{I}$, and hence $\mathbf{A}$ is involutory.

- We need to show that $\frac{1}{2}(\mathbf{I}+\mathbf{A})=\frac{1}{2}\left[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}+\begin{pmatrix}3 & -4\\ 2 & -3\end{pmatrix}\right]=\begin{pmatrix}2 & -2\\ 1 & -1\end{pmatrix}$ is idempotent.

  We have $\begin{pmatrix}2 & -2\\ 1 & -1\end{pmatrix}\begin{pmatrix}2 & -2\\ 1 & -1\end{pmatrix}=\begin{pmatrix}2 & -2\\ 1 & -1\end{pmatrix}$. Hence $\frac{1}{2}(\mathbf{I}+\mathbf{A})$ is idempotent.

  Similarly, $\frac{1}{2}(\mathbf{I}-\mathbf{A})=\frac{1}{2}\left[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}-\begin{pmatrix}3 & -4\\ 2 & -3\end{pmatrix}\right]=\begin{pmatrix}-1 & 2\\ -1 & 2\end{pmatrix}$

  As $\begin{pmatrix}-1 & 2\\ -1 & 2\end{pmatrix}\begin{pmatrix}-1 & 2\\ -1 & 2\end{pmatrix}=\begin{pmatrix}-1 & 2\\ -1 & 2\end{pmatrix}$, then $\frac{1}{2}(\mathbf{I}-\mathbf{A})$ is also idempotent.

♦ Finally, $(\mathbf{I}+\mathbf{A})(\mathbf{I}-\mathbf{A})=\begin{pmatrix}4 & -4\\ 2 & -2\end{pmatrix}\begin{pmatrix}-2 & 4\\ -2 & 4\end{pmatrix}=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}=\mathbf{0}$