MST 121: Resource Material for Chapter B1, Modelling with sequences

1. (a) Use the formula $\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$ to calculate the sum of the first 1000 positive integers.

   (b) By applying this formula one more time, deduce the sum of the integers from 800 to 1000 (inclusive).

2. The table below shows lower bounds for the world's population for the years between 1900 and 1950.

Lower bounds for the world's population from 1900 to 1950						
Year	1900	1910	1920	1930	1940	1950
Year number	0	10	20	30	40	50
Lower bound (in millions)	1550	1750	1860	2070	2300	2400

Source: http://www.census.gov/ipc/www/worldhis.html

   (a) Assuming that this population can be modelled exactly over the intervening period by an exponential model, find to two significant figures the value of the annual proportionate growth rate, *r*. (Use the population measured in millions.)

   (b) Write down the corresponding formula for a lower bound for the world's population P_n (in millions) at *n* years after 1900.

   Throughout the remainder of this question you should use the rounded value of *r* that you gave as the solution to part (a). (This approach is consistent with the solutions given to Example 2.2 and Activity 2.2. in the course text.)

   (c) Use the predictions from the model to complete the table below, **giving your answers to the same degree of accuracy as the given data**.

Lower bounds for the world's population from 1900 to 1950						
Year	1900	1910	1920	1930	1940	1950
Year number	0	10	20	30	40	50
Lower bound (in millions)	1550	1750	1860	2070	2300	2400
Model's prediction						
Percentage error						

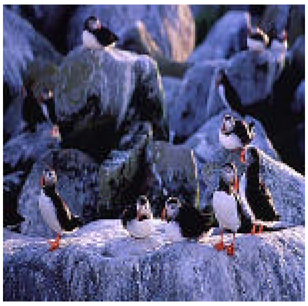
   Use your results to comment on how well the model's output agrees with the given data values.

   (d) After how many years, according to the model, does the world population first reach 3 billion?

   (e) Assuming that this model continues to hold after 1950, what is the world population predicted to be in the year 2001?

   In fact, a lower bound for the world's population in 2001 has been given as 6150 million. Determine the percentage error in your prediction and comment on your result.

3. A population of puffins living on Machias Seal Island off New Brunswick on the Canadian coast was observed to number almost 500 in 1960 and almost 2000 in 1985. Observations of the birth and death rates of the puffins suggest that the following information describes the behaviour of the population during this period:



- The annual proportionate death rate is constant at 0.23
- The annual proportionate birth rate decreases linearly with the population size, P, according to the formula $0.47 - 1.2\times10^{-4}P$

Given that the population size on the 1st January 1960 was 485

(a) Find a recurrence system for P_n, the population size of puffins n years after 1st January 1960.

(b) Show that the recurrence relation obtained is logistic by identifying the values of the parameters r and E from the equation $P_{n+1} - P_n = rP_n\left(1 - \frac{P_n}{E}\right)$

4. The table below shows the number of new cases of AIDS reported in the USA between 1981 and 1996.

Spread of AIDS in the USA from 1981 to 1996						
Year	1981	1982	1983	1984	1985	1986
New cases	422	1614	4749	11055	22996	42255
Year	1987	1988	1989	1990	1991	1992
New cases	71176	106994	149902	198466	257750	335211
Year	1993	1994	1995	1996		
New cases	411887	478756	534806	548102		

Source: http://www.aegis.com

(a) Plot this data on a scatter diagram and convince yourself that it follows the general shape of a population growing according to the logistic model.

(b) Taking 1981 as year 0, use the data from 1981 to 1986 to calculate the value of r, the growth rate at relatively low population levels.

(c) At higher population levels, this value of the growth rate is no longer appropriate. Use the population data from 1994 to 1996 to give an estimate for the annual growth rate centred on 1995.

(d) Assuming that the behaviour of this population satisfies the logistic recurrence relation $P_{n+1} - P_n = rP_n\left(1 - \frac{P_n}{E}\right)$, use your values of r, P_n and $P_{n+1} - P_n$ to estimate the value of the equilibrium population level, E.

(e) Comment on the validity of the assumption made in (d) and the corresponding value of E, with particular reference to the population being modelled.

5. **Mlawula Nature Reserve** is located in north-eastern Swaziland, between latitudes 26°09'E and 26°20'E and between longitudes 31°56'S and 32°06'S covering an area of approximately 16,500 hectares.

In 1971 a population of black wildebeest was introduced to this nature reserve and its growth subsequently monitored. If we let P represent the size of the population at the start of a given year, it was found that:

(i) The annual proportionate birth rate is 1.82 when $P = 250$ and 1.75 when $P = 500$.

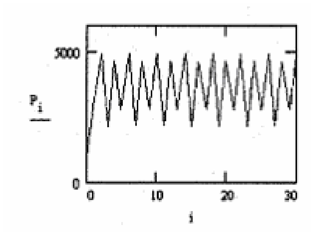
(ii) The annual proportionate death rate is 1.25 when $P = 250$ and 1.38 when $P = 500$.

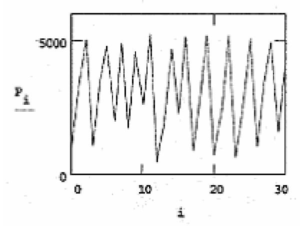
(a) Assuming that the behaviour of this population satisfies the logistic model, estimate the values of the equilibrium population level, E, and proportionate growth rate for low population sizes, r. How would you expect a population characterised by these values of E and r to grow?

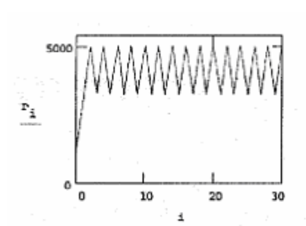
(b) Given that there were initially 50 black wildebeest introduced into the reserve, use the recurrence equation $P_{n+1} - P_n = rP_n\left(1 - \frac{P_n}{E}\right)$ to generate predictions for the population levels over the next 10 years and plot your results on a scatter graph. Is the shape of this graph consistent with your predictions in (a)?

Question six has been reproduced from the year 2000 examination paper **:**

6. Mathcad was used to plot the graphs below, which are the outcomes from using the logistic recurrence relation $P_{n+1} - P_n = rP_n\left(1 - \frac{P_n}{E}\right)$ with the same values of P_0 and E in each case, but with three different values of r.







(a) Estimate the starting value, P_0, and the value of E.

(b) Beside each picture, write down a value of r that could produce the graph, with a brief justification of your answer. (You should assume that each graph is typical of the long-term behaviour.)

7. Suppose that a sequence x_n is generated by the recurrence relation

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{20}{x_n}\right) \quad n = 0,1,2,\ldots$$

To what limit values might such a sequence x_n converge?

8. For each of the sequences below, decide whether or not it converges and, if it does, to what limit. In each case $n = 1, 2, 3, \ldots$

$$(a)\ a_n = \frac{5}{3n-7} \quad (b)\ a_n = n^2 + 7n - 1 \quad (c)\ a_n = \frac{300}{5 + 17(0.4)^n}$$

$$(d)\ a_n = 5 - (-1)^{n-1} \quad (e)\ a_n = \frac{7 + 3n^2}{2 - n^3} \quad (f)\ a_n = \frac{1}{1 + n + n^2 + n^3}$$

9. What is the sum of the following infinite series

$$(a)\ 5 + \frac{5}{3} + \frac{5}{9} + \frac{5}{27} + \frac{5}{81} + \ldots \quad (b)\ 1 - \frac{1}{4} + \frac{1}{16} - \frac{1}{64} + \frac{1}{256} - \ldots \quad (c)\ \sum_{i=1}^{\infty}\left[2 \times \left(-\frac{1}{5}\right)^{i-1}\right]$$

10. (a) If the sum to infinity of a geometric series is twice the first term, find the common ratio.

(b) An infinite geometric series is such that the sum of all the terms after the n^{th} is equal to twice the n^{th} term. Show that the sum to infinity of the complete series is three times the first term.

11. Express each of the following infinite decimals as exact fractions in their lowest terms:

$$(a)\ 0.88888\ldots \quad (b)\ 0.76666666\ldots \quad (c)\ 0.6\dot{4} \quad (d)\ 0.46262626262\ldots$$

$$(e)\ 0.8\dot{2}\dot{6} \quad (f)\ 0.46\dot{1}\dot{2}$$

Answers:

1. (a) $\sum_{i=1}^{1000} i = \frac{1}{2}\times 1000\times 1001 = 500500$

(b) $\sum_{i=800}^{1000} i = \sum_{i=1}^{1000} i - \sum_{i=1}^{799} i = 500500 - \frac{1}{2}\times 799\times 800 = 180900$

2. (a) If we take the year 1900 as $n=0$, then the year 1950 is given by $n=50$. Hence $P_0 = 1550$ and $P_{50} = 2400$ (both in millions). Now by equation (2.2) on page 22 of the text we know that $P_n = (1+r)^n P_0$. Hence $1550(1+r)^{50} = 2400$ from which $(1+r)^{50} \approx 1.55$. It follows that $1+r = \sqrt[50]{1.55}$ and that $r \approx 0.0088$ (which is 0.88%).

(b) $P_n = 1550\times(1.0088)^n$

(c)

Lower bounds for the world's population from 1900 to 1950						
Year	1900	1910	1920	1930	1940	1950
Year number	0	10	20	30	40	50
Lower bound (in millions)	1550	1750	1860	2070	2300	2400
Model's prediction	1550	1690	1850	2020	2200	2400
Percentage error	0	3.4	0.5	2.4	4.3	0

So within the range of the given data, the model provides a reasonably good fit to the given data.

(d) As 1 billion $= 10^9$, and we are working in millions, we need to solve the equation $1550\times(1.0088)^n = 3000$. This gives $(1.0088)^n \approx 1.935$, from which $n = \frac{\ln 1.935}{\ln 1.0088} \approx 75.4$, or approximately 75 years.

(e) $P_{101} = 1550\times 1.0088^{101} \approx 3760$ million, which is in error from the quoted figure by approximately 39%. This shows the danger of extrapolating outside the range of the given data, and serves to emphasise that the predictions made from such calculations must be considered with extreme caution.

3. (a) The proportionate growth rate $R(P)$ at population size P is given by the proportionate birth rate minus the proportionate death rate. i.e.

$$R(P) = (0.47 - 1.2\times 10^{-4} P) - 0.23 = 0.24 - 1.2\times 10^{-4} P$$

And so for the year which starts at time n, $R(P_n) = 0.24 - 1.2\times 10^{-4} P_n$

This is the factor by which the population in year n must be multiplied in order to determine the change in the population during that year. Hence

$P_{n+1} = P_n + \left(0.24 - 1.2\times 10^{-4} P_n\right)\cdot P_n$, and so the recurrence system is

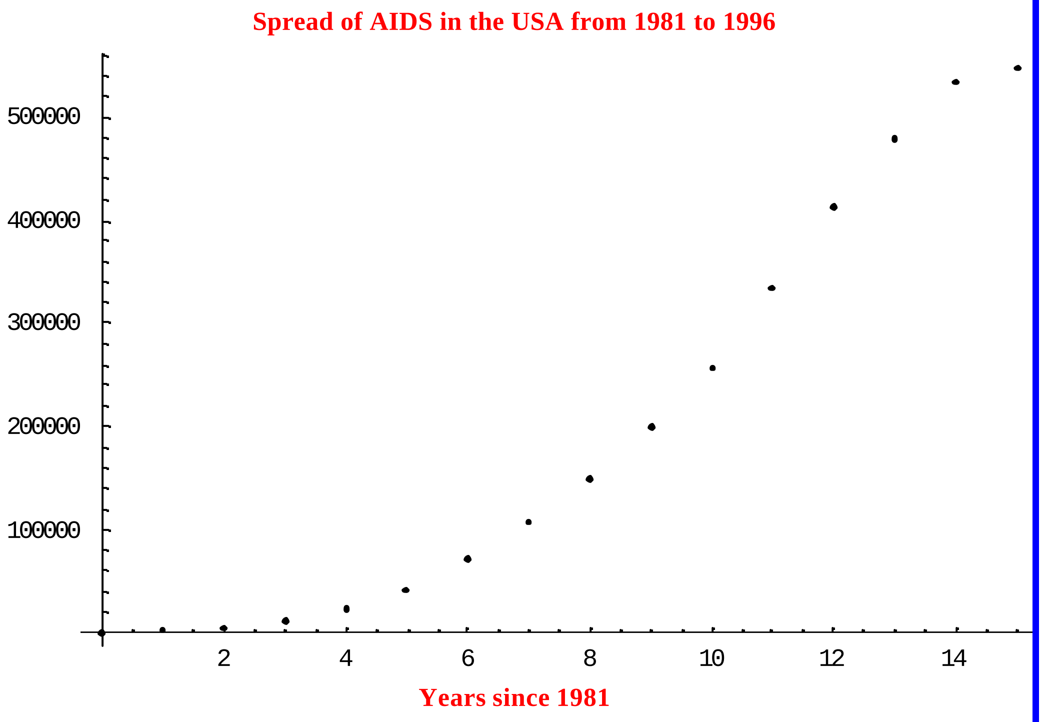
$$P_0 = 485,\ \ P_{n+1} - P_n = \left(0.24 - 1.2\times 10^{-4} P_n\right)\cdot P_n,\ \ n = 0,1,2,\ldots$$

(b) To show that this recurrence relation is logistic, we must show that the right hand side can be written in the form $rP_n\left(1-\frac{P_n}{E}\right)$. We have

$$\left(0.24-1.2\times10^{-4}P_n\right)\cdot P_n = 0.24\left[1-\frac{1.2\times10^{-4}}{0.24}P_n\right]P_n$$
$$=0.24\left[1-5\times10^{-4}P_n\right]P_n$$
$$=0.24\left[1-\frac{P_n}{2000}\right]P_n$$

Which is of the required form, with $r=0.24$ and $E=2000$

4. (a)



(b) With $P_0=422$ and $P_5=42255$, we need to solve $422(1+r)^5=42255$. This gives $1+r=\sqrt[5]{100.13}$ from which $r\approx1.51$.

(c) The size of the population was 478756 in 1994, 534806 in 1995 and 548102 in 1996. Hence there was a growth of 69346 over a three year period with midpoint 1995. This suggests an annual growth rate of approximately 23115 for a population of size 534806.

(d) The logistic recurrence relation is $P_{n+1}-P_n=rP_n\left(1-\frac{P_n}{E}\right)$ where $r\approx1.51$ from consideration of the proportionate growth rate at low population levels. As $P_{n+1}-P_n$ is estimated to be 23115 for the population size $P_n=534806$, this equation becomes $1.51\times534806\left(1-\frac{534806}{E}\right)=23115$.

Hence $1 - \frac{534806}{E} \approx 0.0286$, from which $E \approx 550565$ (which we would sensibly round to 550000.)

(e) Were the population to continue to grow logistically, as indicated by the scatter graph, it would tend toward an equilibrium level given approximately by the value of *E* calculated in (d). However, a population of this nature will not have its growth inhibited by lack of food, competition for resources etc,… in the same way that a population of species might. And although the growth rate must eventually slow down, it is unlikely that it will do so in the short term. In fact, in 1998 there were a total of 665357 new cases reported, which is some 17.3% greater than the equilibrium level calculated in (d).

5. (a) When $P = 250$, the proportionate growth rate is $1.82 - 1.25 = 0.57$. And when $P = 500$, the proportionate growth rate is $1.75 - 1.38 = 0.37$

Now by equation (3.2) of the course text we know that $R(P) = r\left(1 - \frac{P}{E}\right)$.

Substituting in the values from above gives

For $P = 250,\ 0.57 = r\left(1 - \frac{250}{E}\right) \Rightarrow 0.57 = r - \frac{250r}{E}$ (1)

For $P = 500,\ 0.37 = r\left(1 - \frac{500}{E}\right) \Rightarrow 0.37 = r - \frac{500r}{E}$ (2)

If we now subtract equation (2) from $2 \times (1)$ we get $r = 0.77$. Substituting this value of *r* back into equation (1) gives

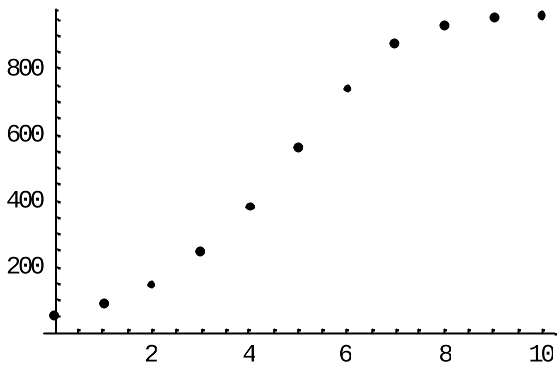
$$0.57 = 0.77 - \frac{250 \times 0.77}{E} \Rightarrow \frac{192.5}{E} = 0.2 \text{ from which } E \approx 963$$

As $0 < r \leq 1$, we would expect the population to converge to *E* (963), with population values always just below *E*. [See table on page 40 of B1]

(b) If you have a graphical calculator, the easiest way to generate the successive population values is to apply the following key sequence:

50 **ENTER** **+** 0.77 **´** **ANS** **´** **(** 1 **–** **ANS** **¸** 963 **)** **ENTER** **ENTER** **ENTER** …

This gives 87, 147, 243, 383, 561, 741, 873, 936, 956, 961,…..



Which is precisely the (logistic) shape predicted by the theory.

6. (a) $P_0 = 1000$ and $E = 4000$

(b) The leftmost graph shows a 4-cycle, with two values above E and two values below E. This will have been generated by a value of r in the range $2.45 \le r < 2.54$, so taking $r = 2.5$ will be about right.

The graph in the centre shows bounded chaotic variation. This will have been generated by a value of r in the range $2.6 \le r \le 3$, so taking $r = 2.8$ will be about right.

The rightmost graph shows a 2-cycle, with one value above E and the other value below E. This will have been generated by a value of r in the range $2 < r \le 2.44$, so taking $r = 2.22$ will be about right.

7. Suppose that $x_n = c$ is a constant sequence generated by the given recurrence relation. Then we must have $x_{n+1} = x_n = c$, and so substituting for x_n and x_{n+1} into the given recurrence relation yields $c = \frac{1}{2}\left(c + \frac{20}{c}\right)$.

Multiplying both sides of this equation by 2 gives $2c = c + \frac{20}{c}$. Hence $c = \frac{20}{c}$ from which $c^2 = 20$ and $c = \pm 2\sqrt{5}$. These are the only two possible limit values for any sequence generated by the recurrence relation.

8. (a) As $n \to \infty$, $3n - 7$ becomes arbitrarily large. Hence $\frac{5}{3n-7}$ becomes arbitrarily small and we have $\lim_{n\to\infty}\left(\frac{5}{3n-7}\right) = 0$.

(b) As $n \to \infty$, $n^2 + 7n$ becomes arbitrarily large. Hence the sequence $a_n = n^2 + 7n - 1$ diverges

(c) As $n \to \infty$, $17(0.4)^n \to 0$. Hence $\lim_{n\to\infty}\left(\frac{300}{5 + 17(0.4)^n}\right) = \frac{300}{5} = 60$

(d) As $(-1)^{n-1}$ oscillates between -1 (n even) and $+1$ (n odd), then the sequence generated by $a_n = 5 - (-1)^{n-1}$ is $4,6,4,6,4,6,4,6,\ldots$. Hence a_n generates a 2-cycle and does not converge.

(e) If we divide both numerator and denominator of $\frac{7+3n^2}{2-n^3}$ by n^3 (the highest power of n), we have

$$\lim_{n\to\infty}\left(\frac{7+3n^2}{2-n^3}\right) = \lim_{n\to\infty}\left(\frac{\frac{7}{n^3}+\frac{3n^2}{n^3}}{\frac{2}{n^3}-\frac{n^3}{n^3}}\right) = \lim_{n\to\infty}\left(\frac{\frac{7}{n^3}+\frac{3}{n}}{\frac{2}{n^3}-1}\right) = 0$$

(f) As $1 + n + n^2 + n^3$ becomes arbitrarily large as $n \to \infty$, $\lim_{n\to\infty}\left(\frac{1}{1+n+n^2+n^3}\right) = 0$

9. (a) $5+\frac{5}{3}+\frac{5}{9}+\frac{5}{27}+\frac{5}{81}+...$ is a geometric series with $a=5$ and $r=\frac{1}{3}$. Hence

$$S_\infty=\frac{a}{1-r}=\frac{5}{1-\frac{1}{3}}=\frac{15}{2}$$

(b) $1-\frac{1}{4}+\frac{1}{16}-\frac{1}{64}+\frac{1}{256}-...$ is a geometric series with $a=1$ and $r=-\frac{1}{4}$. Hence

$$S_\infty=\frac{a}{1-r}=\frac{1}{1-(-\frac{1}{4})}=\frac{4}{5}$$

(c) $\sum_{i=1}^{\infty}\left[2\times\left(-\frac{1}{5}\right)^{i-1}\right]=2-\frac{2}{5}+\frac{2}{25}-...$ is a geometric series with $a=2$ and $r=-\frac{1}{5}$.

Hence $S_\infty=\frac{a}{1-r}=\frac{2}{1-(-\frac{1}{5})}=\frac{5}{3}$

10. Let the first term of the geometric series be a and the common ratio r. Then

(a) $s_\infty=2a\Rightarrow\frac{a}{1-r}=2a$. Hence $1-r=\frac{1}{2}$, from which $r=\frac{1}{2}$

(b) The sum of all the terms after the n^{th} term is given by

$$s_\infty-s_n=\frac{a}{1-r}-\frac{a(1-r^n)}{1-r}=\frac{a-a(1-r^n)}{1-r}=\frac{ar^n}{1-r}$$

If this is to be equal to twice the n^{th} term, then $\frac{ar^n}{1-r}=2ar^{n-1}$

We can now divide both sides by ar^{n-1} to give $\frac{r}{1-r}=2$, from which $r=\frac{2}{3}$

When $r=\frac{2}{3}$, $s_\infty=\frac{a}{1-\frac{2}{3}}=\frac{a}{1/3}=3a=3\times$ the first term

11. (a) $\frac{8}{9}$

(b) Let $x=0.76666...$

Then $10x=7.66666...$ and $100x=76.66666.....$

$\therefore 90x=69$, and $x=\frac{69}{90}=\frac{23}{30}$

(c) Let $x=0.644444...$

Then $10x=6.44444...$ and $100x=64.44444.....$

$\therefore 90x=58$, and $x=\frac{58}{90}=\frac{29}{45}$

(d) Let $x = 0.46262626...$

Then $100x = 46.262626...$ and $10000x = 4626.2626.....$

$\therefore 9900x = 4580$, and $x = \dfrac{4580}{9900} = \dfrac{229}{495}$

(e) Let $x = 0.8262626...$

Then $10x = 8.262626...$ and $1000x = 826.2626.....$

$\therefore 990x = 818$, and $x = \dfrac{818}{990} = \dfrac{409}{495}$

(f) Let $x = 0.46121212...$

Then $100x = 46.121212...$ and $10000x = 4612.121212.....$

$\therefore 9900x = 4566$, and $x = \dfrac{4566}{9900} = \dfrac{761}{1650}$