MS221 : Supplementary Resource Material for Chapter C2

There are no questions on the (revision) material introduced in section one of this Chapter, as all the topics have been comprehensively dealt with in the resource material for Chapter C2 of MST121.

Question 1

Use integration by parts to evaluate each of the following indefinite integrals:

(a) $\int x\sec^2 x\,dx$ (b) $\int x^n \ln x\,dx$ (c) $\int x^2 e^{-2x} dx$

(d) $\int \sin^{-1} x\,dx$ (e) $\int \cos^{-1} x\,dx$ (f) $\int e^{\alpha x}\sin \beta x\,dx$

Question 2

Use integration by parts to evaluate each of the following definite integrals:

(a) $\int_0^{\pi/6} x\sin^2 x\,dx$ (b) $\int_0^1 x\tan^{-1} x\,dx$ (c) $\int_0^{1/k} x^2 e^{kx} dx$

Question 3

Use the suggested substitution to evaluate each of the following integrals:

(a) $\int \sec^2 x \cdot \ln(\tan x)\,dx$ by taking $u = \tan x$ and then integrating by parts.

(b) (i) Show that $\sin^7 x = \sin x\left(1-\cos^2 x\right)^3$

(ii) Hence use the substitution $u = \cos x$ to determine the integral $\int \sin^7 x\,dx$

(c) Use the substitution $u = 1 - t^2$ to show that $\int \frac{1+t^2}{1-t^2}\,dt = \ln\left(\frac{t+1}{t-1}\right) - t + c$.

(d) Use the substitution $u = \frac{1}{x}$ to evaluate $\int_{2/\sqrt{3}}^{2} \frac{1}{x\sqrt{x^2-1}}\,dx$.

(e) Use the substitution $u = \frac{1}{x}$ to show that $\int_e^{e^2} \frac{1}{x\ln x}\,dx = \ln 2$

(f) Use the substitution $u = \frac{1}{x}$ to show that $\int_e^{e^2} \frac{\ln x}{x}\,dx = \frac{3}{2}$

Question 4

Use the suggested substitution to evaluate each of the following integrals:

(a) By taking $u^2 = 1 - x^3$, show that $\int x^5 \sqrt{1-x^3}\, dx = -\frac{2}{45}\left(1-x^3\right)^{3/2}\left(2+3x^3\right)+c$

(b) By taking $x = \dfrac{u^2-5}{1+u^2}$, show that $\displaystyle\int \frac{x\,dx}{\left(5-4x-x^2\right)^{3/2}} = \frac{5-2x}{9\sqrt{5-4x-x^2}}+c$

(c) By taking $u^4 = x$, show that $\displaystyle\int \frac{dx}{x^{1/2} - x^{1/4}} = 2\sqrt{x} + 4\cdot\sqrt[4]{x} + \ln\left(\sqrt[4]{x}-1\right)^4 + c$

(d) By taking $u^2 = 1 + x$, find the exact value of $\displaystyle\int_3^8 \frac{dx}{x\sqrt{1+x}}$

(e) Use the substitution $u^2 = 1 + x^2$ to find the exact value of $\int_1^2 x^3\sqrt{1+x^2}\, dx$

(f) Use the substitution $u^2 = x^2 - 1$ to find the exact value of $\displaystyle\int_{\sqrt{2}}^{\sqrt{5}} \frac{x^3}{\sqrt{x^2-1}}\, dx$

Question 5

Use the suggested substitution to evaluate each of the following integrals:

(a) Use the substitution $x = 3\sin u$ to find $\displaystyle\int \frac{dx}{\sqrt{9-x^2}}$

(b) Use the substitution $t = \tan x$ to find $\displaystyle\int \frac{dx}{4\cos^2 x - 9\sin^2 x}\, dx$

(c) Use the substitution $t = \tan\frac{1}{2}x$ to find $\displaystyle\int \frac{dx}{3+5\cos x}$

(d) By letting $x = \sin\theta$, show that $\displaystyle\int_0^{1/2} \frac{x^4}{\sqrt{1-x^2}}\, dx = \frac{1}{64}\left(4\pi - 7\sqrt{3}\right)$

(e) By letting $x = \tan\theta$, show that $\displaystyle\int_0^1 \frac{dx}{\left(1+x^2\right)^2} = \frac{\pi+2}{8}$

(f) By letting $x = 2\sin t$, show that $\displaystyle\int_1^{\sqrt{3}} \frac{x+3}{\sqrt{4-x^2}}\, dx = \frac{\pi}{2} + \sqrt{3} - 1$

Question 6

Each of the following curves are rotated through 2π radians about the x-axis between the specified limits. Calculate, in cubic units, the volume of the solid of revolution thus formed:

(a) $y = \sin x \cos x$ between $x = 0$ and $x = \frac{\pi}{2}$

(b) $y = \ln x$ between $x = 1$ and $x = e$

(c) $y = \dfrac{x^2 + 1}{x}$ between $x = 5$ and $x = 9$

Question 7

This is a miscellaneous exercise and you may find that some of the integrals are a little trickier than the previous ones.

(a) Use the substitution $x = \sin t$ to show that $\displaystyle\int_0^1 \frac{1}{x + \sqrt{1 - x^2}}\, dx = \frac{\pi}{4}$

(b) Show that $\displaystyle\int \left(1 - x^2\right)^{\frac{1}{2}} dx = \frac{1}{2}\left[\arcsin x + x\sqrt{1 - x^2}\right] + c$

(c) Use the substitution $t = \tan\theta$ to evaluate $\displaystyle\int_{\pi/6}^{\pi/3} \frac{1}{\sin 2\theta}\, d\theta$, giving your answer in the form $p \ln q$.

(d) Use the substitution $t = \tan x$ to evaluate $\displaystyle\int_0^{\pi/4} \frac{1}{3\cos^2 x + \sin^2 x}\, dx$, giving your answer in terms of π.

(e) Show that $\displaystyle\frac{1}{8}\left(\frac{1}{x + 2} + \frac{2 - x}{x^2 + 4}\right) = \frac{1}{(x + 2)(x^2 + 4)}$

Hence prove that $\displaystyle\int_0^2 \frac{1}{(x + 2)(x^2 + 4)}\, dx = \frac{\pi + 2\ln 2}{32}$

(f) (i) Prove that the area of a circle of radius r is πr^2

(ii) Prove that the area enclosed by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is πab

(g) Use the substitution $x = \sin\theta$ to show that $\displaystyle\int_0^{1/2} \frac{x^4}{\sqrt{1 - x^2}}\, dx = \frac{4\pi - 7\sqrt{3}}{64}$

(h) Use the substitution $t = \tan x$ to express $\int_0^{\pi/4} \frac{1}{3\sin 2x + 3}\, dx$ in the form $\frac{p}{q}$, where p and q are coprime integers.

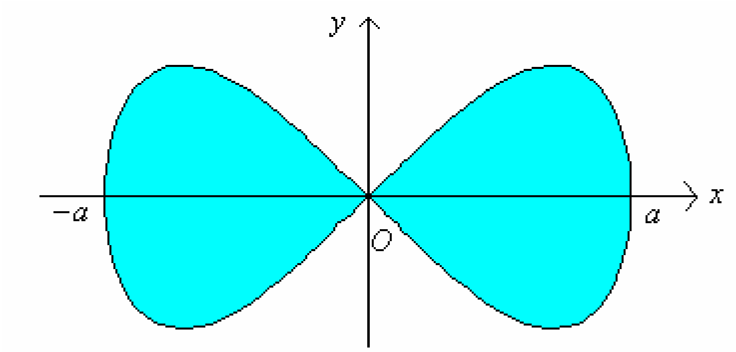
(i) Use the substitution $x = 3\cos^2\theta + 6\sin^2\theta$ to evaluate $\int_3^6 \frac{dx}{\sqrt{(x-3)(6-x)}}\, dx$

(j) Use the substitution $x = \sec^2\theta$ to evaluate $\int_2^5 \frac{dx}{x^2\sqrt{x-1}}$

(k) Use the substitution $x = 4\sin^2\theta$ to show that $\int_0^2 \sqrt{x(4-x)}\, dx = \pi$

(l)
- Express $x^2 + 4x + 3$ in completed square form
- Hence use the substitution $x + 2 = \sec\psi$ to find $\int \frac{1}{\sqrt{x^2 + 4x + 3}}\, dx$

(m) Consider the curve with equation $a^2y^2 = x^2\left(a^2 - x^2\right)$, as shown below.



Show that the total area enclosed by both loops of the curve is $\frac{4}{3}a^2$.

(n) Find $\int \frac{\arcsin x}{\sqrt{1-x^2}}\, dx$

(o) Prove that $\int_1^2 (x-1)^3 \ln x\, dx = \frac{7}{48}$

(p) Given that

$$\int_0^{2\pi/3} \frac{1}{5 + 4\cos x}\, dx = a\pi, \quad a \in \mathbb{Q}$$

Use the substitution $t = \tan\frac{1}{2}x$ to find the value of a.