MS221: Supplementary Resource Material for Chapter B3, Iteration with matrices

1. Use the matrix that represents a scaling with factors 7 and 1 to show algebraically that:

   (a) the point $(0,4)$ is a fixed point of this scaling;

   (b) The x-axis is an invariant line of this scaling

2. Describe any fixed points and invariant lines of each of the following linear transformations

   (a) $r_{\frac{\pi}{4}}$ (b) $q_{\frac{\pi}{3}}$ (c) A scaling with factors -5 and 6 (d) x-shear with factor -2

3. Let f be the linear transformation $f(\mathbf{x}) = \mathbf{A}\mathbf{x}$, where $\mathbf{A} = \begin{pmatrix} -3 & 2 \\ 3 & -4 \end{pmatrix}$. Show that, under f,

   - All points on the line $3x + 2y = 0$ are scaled by a factor of -6 to another point on the same line
   - All points on the line $y = x$ are scaled by a factor of -1 to another point on the same line

4. (a) Write down the characteristic equation of the matrix $\mathbf{A} = \begin{pmatrix} 3 & -1 \\ -9 & -5 \end{pmatrix}$

   (b) By solving the characteristic equation, find the eigenvalues and eigenlines of $\mathbf{A}$.

   (c) For each eigenvalue, give one eigenvector.

5. (a) Use the characteristic equation of the matrix $\mathbf{A} = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix}$ to prove that the eigenvalues of a diagonal matrix are the elements on the diagonal.

   (b) Find the eigenlines of the diagonal matrix $\mathbf{A} = \begin{pmatrix} 7 & 0 \\ 0 & -3 \end{pmatrix}$

6. The flattening represented by the matrix $\mathbf{A} = \begin{pmatrix} 8 & 4 \\ -2 & -1 \end{pmatrix}$ collapses the plane onto the line $x + 4y = 0$.

   (a) Find the eigenvalues of $\mathbf{A}$.

   (b) Deduce that the flattening maps every point of the eigenline $2x + y = 0$ to the origin and that $x + 4y = 0$ is also an eigenline of $\mathbf{A}$.

7. For each matrix below, identify which type of basic linear transformation it represents and state how many eigenlines it should have, based on your geometric understanding of the linear transformation. Verify your answers by finding the eigenvalues and eigenlines for each of these matrices.

   (a) $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (b) $\mathbf{B} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ (c) $\mathbf{C} = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$ (d) $\mathbf{E} = \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}$ (e) $\mathbf{F} = \begin{pmatrix} -2 & 6 \\ 1 & -3 \end{pmatrix}$

8. By formulating the characteristic equation, but not solving it, show that the matrix $\mathbf{A}=\begin{pmatrix}-7 & -8\\ 5 & 2\end{pmatrix}$ has no real eigenvalues.

9. (a) Express the matrix $\mathbf{A}=\begin{pmatrix}2 & 3\\ 5 & 4\end{pmatrix}$ in the form $\mathbf{PDP}^{-1}$ in two different ways, where $\mathbf{D}$ is a diagonal matrix.

(b) In each case, check that expanding your $\mathbf{PDP}^{-1}$ form recovers the original matrix $\mathbf{A}$.

10. (a) Express the matrix $\mathbf{A}=\begin{pmatrix}5 & 3\\ 4 & 4\end{pmatrix}$ in the form $\mathbf{PDP}^{-1}$.

(b) Hence write down $\mathbf{A}^3$, $\mathbf{A}^6$ and $\mathbf{A}^{10}$, each as a single matrix.

(c) Use your answer to (a) to write down an expression for $\mathbf{A}^n$, and hence write down an expression for $\mathbf{A}^n$ as a single matrix. Use this matrix to check your answers to (b).

11. Consider the matrix $\mathbf{A}=\begin{pmatrix}2 & 3\\ 5 & 4\end{pmatrix}$ whose eigenvalues and eigenlines you found in question 9.

(a) Use properties of generalised scalings to *describe* the location of the image P' of the point $P(2,-5)$ under the linear transformation represented by $\mathbf{A}$.

(b) Calculate P' and draw a sketch to show that the locations of P and P' match your description in part (a).

12. Consider the matrix $\mathbf{A}=\begin{pmatrix}-7 & 10\\ 2 & -6\end{pmatrix}$ and the point $(1,1)$ represented by the vector $x_0=\begin{pmatrix}1\\ 1\end{pmatrix}$

(a) Calculate the first five points in the iteration sequence $\mathbf{x}_{n+1}=\mathbf{A}\mathbf{x}_n$ $(n=0,1,2,\ldots)$ with initial point $(1,1)$.

(b) Find a formula in terms of n for the vector $\mathbf{x}_n$ which represents the $(n+1)$th point in this iteration sequence. Use the formula to calculate the tenth point, represented by the vector $\mathbf{x}_9$.

(c) Calculate the ratio $\frac{y_n}{x_n}$ for each of the points (x_n, y_n) found in part (a). Do these points appear to tend to a particular value? If so, explain the significance of this value geometrically.

13. Use the procedure box at the top of page 46 of the course unit to offer an explanation of your findings in question 12 (c).

Answers:

1. The matrix $\begin{pmatrix} 7 & 0 \\ 0 & 1 \end{pmatrix}$ represents the scaling with factors 7 and 1.

(a) $\begin{pmatrix} 7 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$, so the point $(0,4)$ is a fixed point of this scaling.

(b) The image of an arbitrary point $(c,0)$ on the x-axis under this scaling is given by

$\begin{pmatrix} 7 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} c \\ 0 \end{pmatrix} = \begin{pmatrix} 7c \\ 0 \end{pmatrix}$, and $(7c,0)$ lies on the x-axis.

Hence every point on the x-axis has its image on that axis. Also, as c varies, this image ranges over the complete x-axis. Hence the x-axis is an invariant line of this scaling.

2. (a) $r_{\frac{\pi}{4}}$ is a rotation about the origin through $\frac{\pi}{4}$ radians anticlockwise. Hence there is just one fixed point, the centre of rotation $(0,0)$, and no invariant lines.

(b) $q_{\frac{\pi}{3}}$ is a reflection in the line $y = \tan\left(\frac{\pi}{3}x\right)$, or equivalently $y = \sqrt{3}x$.

Hence $y = \sqrt{3}x$ is an invariant line, and so is $y = -\frac{1}{\sqrt{3}}x$ (the line perpendicular to it).

(c) A scaling with factors -5 and 6 has matrix $\mathbf{A} = \begin{pmatrix} -5 & 0 \\ 0 & 6 \end{pmatrix}$

As $\mathbf{A}\begin{pmatrix} c \\ 0 \end{pmatrix} = \begin{pmatrix} -5 & 0 \\ 0 & 6 \end{pmatrix}\begin{pmatrix} c \\ 0 \end{pmatrix} = \begin{pmatrix} -5c \\ 0 \end{pmatrix} = -5\begin{pmatrix} c \\ 0 \end{pmatrix}$ and $\mathbf{A}\begin{pmatrix} 0 \\ c \end{pmatrix} = \begin{pmatrix} -5 & 0 \\ 0 & 6 \end{pmatrix}\begin{pmatrix} 0 \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 6c \end{pmatrix} = 6\begin{pmatrix} 0 \\ c \end{pmatrix}$

It follows that both the x and y-axes are invariant lines under this scaling, and that the origin, $(0,0)$, is the only fixed point.

(d) The matrix representing an x-shear with factor -2 is $\mathbf{A} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$.

Under this shear the x-axis is the only invariant line and all points on it are fixed points.

This follows because $\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} c \\ 0 \end{pmatrix} = \begin{pmatrix} c \\ 0 \end{pmatrix}$, showing that all points on the x-axis are invariant.

3. ♦ Let $\begin{pmatrix} 2c \\ -3c \end{pmatrix}$ be an arbitrary vector on the line $3x + 2y = 0$. Then, under f,

$$\mathbf{A}\begin{pmatrix} 2c \\ -3c \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ 3 & -4 \end{pmatrix}\begin{pmatrix} 2c \\ -3c \end{pmatrix} = \begin{pmatrix} -12c \\ 18c \end{pmatrix} = -6\begin{pmatrix} 2c \\ -3c \end{pmatrix}$$

Hence all points on the line $3x + 2y = 0$ are scaled by a factor of -6 to another point on the same line.

- Let $\begin{pmatrix} c \\ c \end{pmatrix}$ be an arbitrary vector on the line $y = x$. Then, under f,

$$\mathbf{A}\begin{pmatrix} c \\ c \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ 3 & -4 \end{pmatrix}\begin{pmatrix} c \\ c \end{pmatrix} = \begin{pmatrix} -c \\ -c \end{pmatrix} = -\begin{pmatrix} c \\ c \end{pmatrix}$$

  Hence all points on the line $y = x$ are scaled by a factor of -1 to another point on the same line.

4. (a) The characteristic equation is $k^2 - \text{tr}\mathbf{A} \cdot k + \det \mathbf{A} = 0 \Rightarrow k^2 + 2k - 24 = 0$

   (b) $\therefore (k+6)(k-4) = 0$, from which $k = -6, 4$. Hence the eigenvalues of $\mathbf{A}$ are -6 and 4.

   For $k = -6$ the eigenline is found by solving $\begin{pmatrix} 3 & -1 \\ -9 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -6\begin{pmatrix} x \\ y \end{pmatrix}$ as $y = 9x$

   For $k = 4$ the eigenline is found by solving $\begin{pmatrix} 3 & -1 \\ -9 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 4\begin{pmatrix} x \\ y \end{pmatrix}$ as $y = -x$

   (c) $\begin{pmatrix} 1 \\ 9 \end{pmatrix}$ is an eigenvector corresponding to $k = -6$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to $k = 4$.

5. (a) The characteristic equation is $k^2 - \text{tr}\mathbf{A} \cdot k + \det \mathbf{A} = 0 \Rightarrow k^2 - (a_{11} + a_{22})k + a_{11}a_{22} = 0$

   $\therefore (k - a_{11})(k - a_{22}) = 0$ from which $k = a_{11}, a_{22}$

   Hence the eigenvalues of a diagonal matrix are the elements on the diagonal.

   (b) By (a) we know that the eigenvalues of $\mathbf{A} = \begin{pmatrix} 7 & 0 \\ 0 & -3 \end{pmatrix}$ are 7 and -3.

   For $k = 7$, $\begin{pmatrix} 7 & 0 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 7\begin{pmatrix} x \\ y \end{pmatrix}$ gives the eigenline $y = 0$ (the x-axis).

   For $k = -3$, $\begin{pmatrix} 7 & 0 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -3\begin{pmatrix} x \\ y \end{pmatrix}$ gives the eigenline $x = 0$ (the y-axis).

6. (a) The characteristic equation is $k^2 - \text{tr}\mathbf{A} \cdot k + \det \mathbf{A} = 0 \Rightarrow k^2 - 7k = 0$, from which $k = 0, 7$.

   (b) For $k = 0$, $\begin{pmatrix} 8 & 4 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0\begin{pmatrix} x \\ y \end{pmatrix}$ gives the eigenline $2x + y = 0$. Hence every point on the line $2x + y = 0$ is mapped to the origin.

   For $k = 7$, $\begin{pmatrix} 8 & 4 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 7\begin{pmatrix} x \\ y \end{pmatrix}$ gives the eigenline $x + 4y = 0$.

7. (a) $\mathbf{A}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ represents a reflection in the line $y=x$, which makes an angle of $\frac{\pi}{4}$ with the direction of the positive x-axis. It should have two eigenlines.

One is the axis of reflection $y=x$ with corresponding eigenvalue 1, since all points on this line are fixed points.

The other is the line perpendicular to the axis of reflection, $y=-x$ with corresponding eigenvalue -1, as points on this line are mapped to the same line but on the opposite side of the origin.

The characteristic equation is $k^2-\text{tr}\mathbf{A}\cdot k+\det\mathbf{A}=0\Rightarrow k^2-1=0$, from which $k=\pm 1$.

For $k=-1$ the eigenline is found by solving $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=-\begin{pmatrix}x\\ y\end{pmatrix}$ as $y=-x$

And for $k=1$ the eigenline is found by solving $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}x\\ y\end{pmatrix}$ as $y=x$.

Thus confirming our geometric interpretation of the matrix.

(b) $\mathbf{B}=\begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}=\begin{pmatrix}\cos\frac{\pi}{4} & -\sin\frac{\pi}{4}\\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4}\end{pmatrix}$ and so represents $r_{\frac{\pi}{4}}$, a rotation about the origin through $\frac{\pi}{4}$ radians anticlockwise. We would not expect any lines to remain invariant under this transformation, so consequently we would not expect $\begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}$ to have any real eigenvalues.

The characteristic equation is $k^2-\text{tr}\mathbf{A}\cdot k+\det\mathbf{A}=0\Rightarrow k^2-\sqrt{2}k+1=0$. This equation has no real solutions, and consequently the transformation that it represents has no eigenlines, thus confirming our geometric interpretation of the matrix.

(c) $\mathbf{C}=\begin{pmatrix}3 & 0\\ 0 & 4\end{pmatrix}$ represents a scaling with factors 3 and 4. It should have two eigenlines, the x and y-axes, with corresponding eigenvalues 3 and 4, respectively.

As $\mathbf{C}$ is a diagonal matrix, its eigenvalues are 3 and 4.

For $k=3$ the eigenline is found by solving $\begin{pmatrix}3 & 0\\ 0 & 4\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=3\begin{pmatrix}x\\ y\end{pmatrix}$ as $y=0$, which is the x-axis.

For $k=4$ the eigenline is found by solving $\begin{pmatrix}3 & 0\\ 0 & 4\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=4\begin{pmatrix}x\\ y\end{pmatrix}$ as $x=0$, which is the y-axis.

Thus confirming our geometric interpretation of the matrix.

(d) $\mathbf{E}=\begin{pmatrix}1 & 0\\ 5 & 1\end{pmatrix}$ represents a y-shear with factor 5. It should have one eigenline, the y-axis, with corresponding eigenvalue 1 (as points on the y-axis are fixed points for a y-shear.)

The matrix $\begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}$ is a triangular matrix, so its eigenvalues are on the leading diagonal. Hence this matrix has only one eigenvalue, $k = 1$.

For $k = 1$ the eigenline is found by solving $\begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 1\begin{pmatrix} x \\ y \end{pmatrix}$ as $x = 0$, which is the y-axis.

Thus confirming our geometric interpretation of the matrix.

(e) As $\begin{vmatrix} -2 & 6 \\ 1 & -3 \end{vmatrix} = 0$, then $\begin{pmatrix} -2 & 6 \\ 1 & -3 \end{pmatrix}$ represents a flattening of the plane (onto $x + 2y = 0$.)

Hence we would expect $\begin{pmatrix} -2 & 6 \\ 1 & -3 \end{pmatrix}$ to have two eigenlines, one of which is $x + 2y = 0$, and the other having the property that every point on it maps to the origin.

The characteristic equation is $k^2 - \text{tr}\mathbf{A} \cdot k + \det \mathbf{A} = 0 \Rightarrow k^2 + 5k = 0$, from which $k = -5, 0$.

For $k = -5$, $\begin{pmatrix} -2 & 6 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -5\begin{pmatrix} x \\ y \end{pmatrix}$ gives the eigenline $x + 2y = 0$.

For $k = 0$, $\begin{pmatrix} -2 & 6 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0\begin{pmatrix} x \\ y \end{pmatrix}$ gives the eigenline $x - 3y = 0$. As $k = 0$, all points on this eigenline map to the origin.

Thus confirming our geometric interpretation of the matrix.

8. The characteristic equation is $k^2 - \text{tr}\mathbf{A} \cdot k + \det \mathbf{A} = 0 \Rightarrow k^2 + 5k + 26 = 0$.

For this quadratic equation, $b^2 - 4ac = 5^2 - 4 \times 1 \times 26 = -79 < 0$. Hence the characteristic equation has no real roots and so the matrix $\mathbf{A}$ has no real eigenvalues.

9. (a) The characteristic equation is $k^2 - \text{tr}\mathbf{A} \cdot k + \det \mathbf{A} = 0 \Rightarrow k^2 - 6k - 7 = 0$, from which $k = -1, 7$

For $k = -1$, $\begin{pmatrix} 2 & 3 \\ 5 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -\begin{pmatrix} x \\ y \end{pmatrix}$ from which $x + y = 0$ is an eigenline and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ an eigenvector.

For $k = 7$, $\begin{pmatrix} 2 & 3 \\ 5 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 7\begin{pmatrix} x \\ y \end{pmatrix}$ from which $5x - 3y = 0$ is an eigenline and $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$ an eigenvector.

- If we take $k = -1$ and $k = 7$ to occupy the a_{11} and a_{22} positions respectively in the diagonal matrix $\mathbf{D}$, then we must have the eigenvectors $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$ in columns 1 and 2 respectively of the matrix $\mathbf{P}$.

Hence $\mathbf{D}=\begin{pmatrix}-1 & 0\\ 0 & 7\end{pmatrix}$, $\mathbf{P}=\begin{pmatrix}1 & 3\\ -1 & 5\end{pmatrix}$ and with this choice of $\mathbf{P}$ we have $\mathbf{P}^{-1}=\begin{pmatrix}\frac{5}{8} & -\frac{3}{8}\\ \frac{1}{8} & \frac{1}{8}\end{pmatrix}$.

$$\therefore\begin{pmatrix}2 & 3\\ 5 & 4\end{pmatrix}=\mathbf{PDP}^{-1}=\begin{pmatrix}1 & 3\\ -1 & 5\end{pmatrix}\begin{pmatrix}-1 & 0\\ 0 & 7\end{pmatrix}\begin{pmatrix}\frac{5}{8} & -\frac{3}{8}\\ \frac{1}{8} & \frac{1}{8}\end{pmatrix}$$

Alternatively we may reverse the order of the columns in the $\mathbf{P}$ matrix and interchange the position of the eigenvalues in the $\mathbf{D}$ matrix.

This gives $\mathbf{D}=\begin{pmatrix}7 & 0\\ 0 & -1\end{pmatrix}$, $\mathbf{P}=\begin{pmatrix}3 & 1\\ 5 & -1\end{pmatrix}$ and with this choice of $\mathbf{P}$ we have $\mathbf{P}^{-1}=\begin{pmatrix}\frac{1}{8} & \frac{1}{8}\\ \frac{5}{8} & -\frac{3}{8}\end{pmatrix}$.

$$\therefore\begin{pmatrix}2 & 3\\ 5 & 4\end{pmatrix}=\mathbf{PDP}^{-1}=\begin{pmatrix}3 & 1\\ 5 & -1\end{pmatrix}\begin{pmatrix}7 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\frac{1}{8} & \frac{1}{8}\\ \frac{5}{8} & -\frac{3}{8}\end{pmatrix}$$

(b)
$$\begin{aligned}\begin{pmatrix}1 & 3\\ -1 & 5\end{pmatrix}\begin{pmatrix}-1 & 0\\ 0 & 7\end{pmatrix}\begin{pmatrix}\frac{5}{8} & -\frac{3}{8}\\ \frac{1}{8} & \frac{1}{8}\end{pmatrix}&=\frac{1}{8}\begin{pmatrix}1 & 3\\ -1 & 5\end{pmatrix}\begin{pmatrix}-1 & 0\\ 0 & 7\end{pmatrix}\begin{pmatrix}5 & -3\\ 1 & 1\end{pmatrix}\\ &=\frac{1}{8}\begin{pmatrix}1 & 3\\ -1 & 5\end{pmatrix}\begin{pmatrix}-5 & 3\\ 7 & 7\end{pmatrix}\\ &=\frac{1}{8}\begin{pmatrix}16 & 24\\ 40 & 32\end{pmatrix}\\ &=\begin{pmatrix}2 & 3\\ 5 & 4\end{pmatrix}\end{aligned}$$

$$\begin{aligned}\begin{pmatrix}3 & 1\\ 5 & -1\end{pmatrix}\begin{pmatrix}7 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\frac{1}{8} & \frac{1}{8}\\ \frac{5}{8} & -\frac{3}{8}\end{pmatrix}&=\frac{1}{8}\begin{pmatrix}3 & 1\\ 5 & -1\end{pmatrix}\begin{pmatrix}7 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1 & 1\\ 5 & -3\end{pmatrix}\\ &=\frac{1}{8}\begin{pmatrix}3 & 1\\ 5 & -1\end{pmatrix}\begin{pmatrix}7 & 7\\ -5 & 3\end{pmatrix}\\ &=\frac{1}{8}\begin{pmatrix}16 & 24\\ 40 & 32\end{pmatrix}\\ &=\begin{pmatrix}2 & 3\\ 5 & 4\end{pmatrix}\end{aligned}$$

10. The characteristic equation is $k^2-\text{tr}\mathbf{A}\cdot k+\det\mathbf{A}=0\Rightarrow k^2-9k+8=0$, from which $k=1,8$.

For $k=1$, $\begin{pmatrix}5 & 3\\ 4 & 4\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}x\\ y\end{pmatrix}$ from which $4x+3y=0$ is an eigenline and $\begin{pmatrix}3\\ -4\end{pmatrix}$ an eigenvector.

For $k=8$, $\begin{pmatrix}5 & 3\\ 4 & 4\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=8\begin{pmatrix}x\\ y\end{pmatrix}$ from which $x-y=0$ is an eigenline and $\begin{pmatrix}1\\ 1\end{pmatrix}$ an eigenvector.

If we take $k=1$ and $k=8$ to occupy the a_{11} and a_{22} positions respectively in the diagonal matrix $\mathbf{D}$, then we must have the eigenvectors $\begin{pmatrix}3\\-4\end{pmatrix}$ and $\begin{pmatrix}1\\1\end{pmatrix}$ in columns 1 and 2 respectively of the matrix $\mathbf{P}$.

This gives $\mathbf{D}=\begin{pmatrix}1&0\\0&8\end{pmatrix}$, $\mathbf{P}=\begin{pmatrix}3&1\\-4&1\end{pmatrix}$ and with this choice of $\mathbf{P}$ we have $\mathbf{P}^{-1}=\begin{pmatrix}\frac{1}{7}&-\frac{1}{7}\\\frac{4}{7}&\frac{3}{7}\end{pmatrix}$

$$\therefore \mathbf{A}=\mathbf{PDP}^{-1}=\begin{pmatrix}5&3\\4&4\end{pmatrix}=\begin{pmatrix}3&1\\-4&1\end{pmatrix}\begin{pmatrix}1&0\\0&8\end{pmatrix}\begin{pmatrix}\frac{1}{7}&-\frac{1}{7}\\\frac{4}{7}&\frac{3}{7}\end{pmatrix}=\frac{1}{7}\begin{pmatrix}3&1\\-4&1\end{pmatrix}\begin{pmatrix}1&0\\0&8\end{pmatrix}\begin{pmatrix}1&-1\\4&3\end{pmatrix}$$

(b) $\therefore \mathbf{A}^n=\mathbf{PD}^n\mathbf{P}^{-1}=\frac{1}{7}\begin{pmatrix}3&1\\-4&1\end{pmatrix}\begin{pmatrix}1&0\\0&8\end{pmatrix}^n\begin{pmatrix}1&-1\\4&3\end{pmatrix}=\frac{1}{7}\begin{pmatrix}3&1\\-4&1\end{pmatrix}\begin{pmatrix}1&0\\0&8^n\end{pmatrix}\begin{pmatrix}1&-1\\4&3\end{pmatrix}$. Hence

$$\mathbf{A}^3=\frac{1}{7}\begin{pmatrix}3&1\\-4&1\end{pmatrix}\begin{pmatrix}1&0\\0&8^3\end{pmatrix}\begin{pmatrix}1&-1\\4&3\end{pmatrix}=\frac{1}{7}\begin{pmatrix}3&1\\-4&1\end{pmatrix}\begin{pmatrix}1&-1\\2048&1536\end{pmatrix}=\frac{1}{7}\begin{pmatrix}2051&1533\\2044&1540\end{pmatrix}=\begin{pmatrix}293&219\\292&220\end{pmatrix}$$

$$\mathbf{A}^4=\frac{1}{7}\begin{pmatrix}3&1\\-4&1\end{pmatrix}\begin{pmatrix}1&0\\0&8^4\end{pmatrix}\begin{pmatrix}1&-1\\4&3\end{pmatrix}=\frac{1}{7}\begin{pmatrix}3&1\\-4&1\end{pmatrix}\begin{pmatrix}1&-1\\16384&12288\end{pmatrix}=\begin{pmatrix}2341&1755\\2340&1756\end{pmatrix}$$

$$\mathbf{A}^{10}=\frac{1}{7}\begin{pmatrix}3&1\\-4&1\end{pmatrix}\begin{pmatrix}1&0\\0&8^{10}\end{pmatrix}\begin{pmatrix}1&-1\\4&3\end{pmatrix}=\begin{pmatrix}613566757&460175067\\613566756&460175068\end{pmatrix}$$

(c) $\therefore \mathbf{A}^n=\frac{1}{7}\begin{pmatrix}3&1\\-4&1\end{pmatrix}\begin{pmatrix}1&0\\0&8^n\end{pmatrix}\begin{pmatrix}1&-1\\4&3\end{pmatrix}=\frac{1}{7}\begin{pmatrix}3&1\\-4&1\end{pmatrix}\begin{pmatrix}1&-1\\4\cdot8^n&3\cdot8^n\end{pmatrix}=\begin{pmatrix}\frac{1}{7}\left(3+4\cdot8^n\right)&\frac{1}{7}\left(3\cdot8^n-3\right)\\\frac{1}{7}\left(4\cdot8^n-4\right)&\frac{1}{7}\left(4+3\cdot8^n\right)\end{pmatrix}$

$$\therefore \mathbf{A}^3=\begin{pmatrix}\frac{1}{7}\left(3+4\cdot8^3\right)&\frac{1}{7}\left(3\cdot8^3-3\right)\\\frac{1}{7}\left(4\cdot8^3-4\right)&\frac{1}{7}\left(4+3\cdot8^3\right)\end{pmatrix}=\begin{pmatrix}293&219\\292&220\end{pmatrix}$$

$$\mathbf{A}^4=\begin{pmatrix}\frac{1}{7}\left(3+4\cdot8^4\right)&\frac{1}{7}\left(3\cdot8^4-3\right)\\\frac{1}{7}\left(4\cdot8^4-4\right)&\frac{1}{7}\left(4+3\cdot8^4\right)\end{pmatrix}=\begin{pmatrix}2341&1755\\2340&1756\end{pmatrix}$$

And $\mathbf{A}^4=\begin{pmatrix}\frac{1}{7}\left(3+4\cdot8^{10}\right)&\frac{1}{7}\left(3\cdot8^{10}-3\right)\\\frac{1}{7}\left(4\cdot8^{10}-4\right)&\frac{1}{7}\left(4+3\cdot8^{10}\right)\end{pmatrix}=\begin{pmatrix}613566757&460175067\\613566756&460175068\end{pmatrix}$

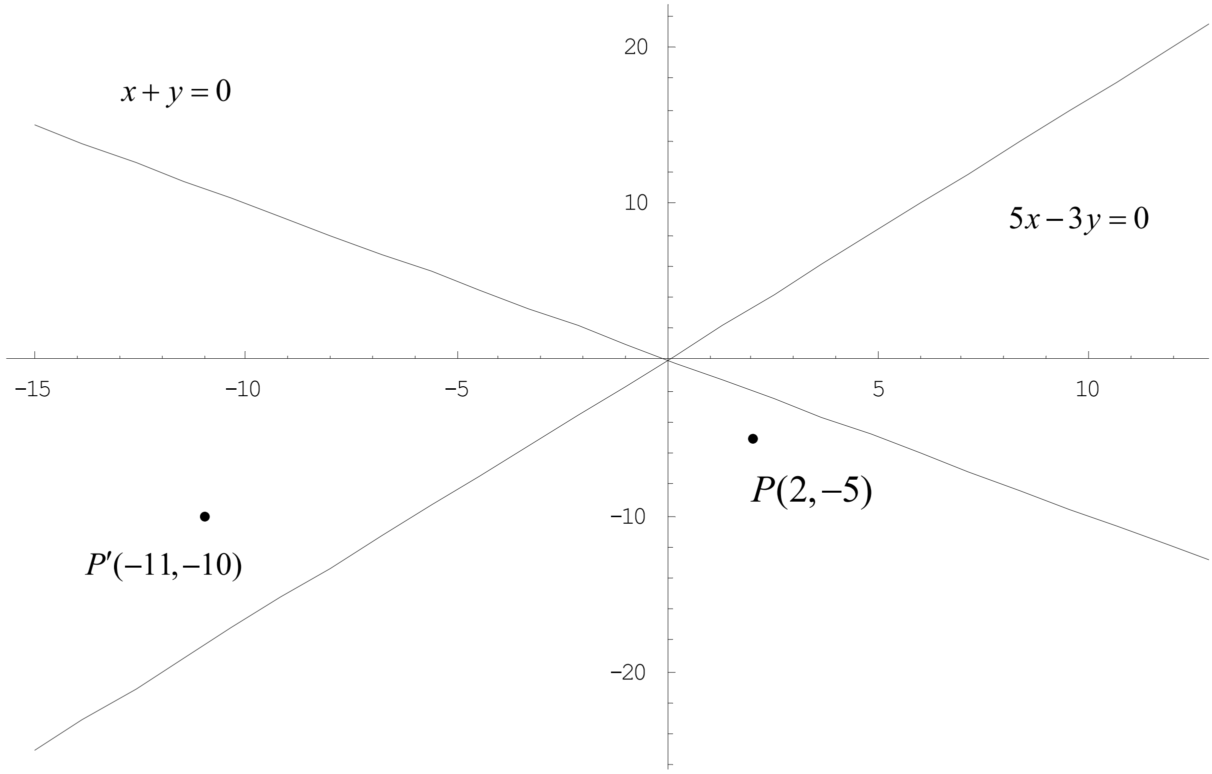
11. (a) We know from question nine that the matrix $\mathbf{A}=\begin{pmatrix}2&3\\5&4\end{pmatrix}$ has eigenlines $k=-1,7$ and corresponding eigenlines $x+y=0$ and $5x-3y=0$. Hence, by property (a) of the "Properties of generalised scalings" [page 36],

As $-1<0$ then P' lies on the opposite side of the line $5x-3y=0$ to P

And since $7>0$ then P' lies on the same side of the line $x+y=0$ to P

Furthermore, by property (b) we know that the distance from P' to the line $5x-3y=0$ is the same as the distance from P to that line and that the distance from P' to the line $x+y=0$ is 7 times the distance from P to that line.

(b) The image, P', is given by $\begin{pmatrix}2 & 3\\5 & 4\end{pmatrix}\begin{pmatrix}2\\-5\end{pmatrix}=\begin{pmatrix}-11\\-10\end{pmatrix}$



Which agrees with our description in part (a).

12. (a) (1,1), (3,−4), (−61,30), (727,−302) and (−8109,3266)

(b) As the closed form for $\mathbf{x}_n$ is $\mathbf{x}_n=\mathbf{A}^n\mathbf{x}_0$, we must firstly write $\mathbf{A}^n$ in the form $\mathbf{PD}^n\mathbf{P}^{-1}$.

The characteristic equation is $k^2-\text{tr}\mathbf{A}\cdot k+\det\mathbf{A}=0 \Rightarrow k^2+13k+22=0$, giving $k=-11,-2$.

For $k=-11$, $\begin{pmatrix}-7 & 10\\2 & -6\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=-11\begin{pmatrix}x\\y\end{pmatrix}$ from which $2x+5y=0$ is an eigenline and $\begin{pmatrix}5\\-2\end{pmatrix}$ an eigenvector.

For $k=-2$, $\begin{pmatrix}-7 & 10\\2 & -6\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=-2\begin{pmatrix}x\\y\end{pmatrix}$ from which $x-2y=0$ is an eigenline and $\begin{pmatrix}2\\1\end{pmatrix}$ an eigenvector.

This gives $\mathbf{D}=\begin{pmatrix}-11 & 0\\0 & -2\end{pmatrix}$, $\mathbf{P}=\begin{pmatrix}5 & 2\\-2 & 1\end{pmatrix}$ and with this choice of $\mathbf{P}$ we have $\mathbf{P}^{-1}=\begin{pmatrix}\frac{1}{9} & -\frac{2}{9}\\\frac{2}{9} & \frac{5}{9}\end{pmatrix}$

$$\therefore \mathbf{A} = \mathbf{PDP}^{-1} = \begin{pmatrix} 5 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} -11 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} \frac{1}{9} & -\frac{2}{9} \\ \frac{2}{9} & \frac{5}{9} \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 5 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} -11 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 2 & 5 \end{pmatrix}$$

$$\therefore \mathbf{A}^n = \mathbf{PD}^n\mathbf{P}^{-1} = \frac{1}{9}\begin{pmatrix} 5 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} (-11)^n & 0 \\ 0 & (-2)^n \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 2 & 5 \end{pmatrix}$$

So with $\mathbf{x}_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, a formula for the closed form for $\mathbf{x}_n$ is given by

$$\begin{aligned} \mathbf{x}_n &= \mathbf{A}^n\mathbf{x}_0 \\ &= \frac{1}{9}\begin{pmatrix} 5 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} (-11)^n & 0 \\ 0 & (-2)^n \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \frac{1}{9}\begin{pmatrix} 5 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} (-11)^n & 0 \\ 0 & (-2)^n \end{pmatrix}\begin{pmatrix} -1 \\ 7 \end{pmatrix} \\ &= \frac{1}{9}\begin{pmatrix} 5 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} -\cdot(-11)^n \\ 7\cdot(-2)^n \end{pmatrix} \\ &= \frac{1}{9}\begin{pmatrix} 14\cdot(-2)^n - 5\cdot(-11)^n \\ 2\cdot(-11)^n + 7\cdot(-2)^n \end{pmatrix} \end{aligned}$$

$\therefore \mathbf{x}_9 = (1309970143, -523988774)$

(c) The ratios have been given in the following table, each correct to 5 significant figures,

n	$\mathbf{x}_n$	x_n	y_n	$\frac{y_n}{x_n}$
0	$\mathbf{x}_0$	1	1	1
1	$\mathbf{x}_1$	3	–4	–1.3333
2	$\mathbf{x}_2$	–61	30	–0.49180
3	$\mathbf{x}_3$	727	–302	–0.41541
4	$\mathbf{x}_4$	–8109	3266	–0.40276
9	$\mathbf{x}_9$	1309970143	–523988774	–0.40000

The ratios appear to be getting closer to -0.4, which is the gradient of the eigenline $2x + 5y = 0$.

13. Let f be the linear transformation represented by the 2×2 matrix $\mathbf{A}=\begin{pmatrix}-7 & 10\\ 2 & -6\end{pmatrix}$. This matrix has two distinct, non-zero eigenvalues, $k_1=-11$ and $k_2=-2$, with corresponding eigenlines $l_1: 2x+5y=0$ and $l_2: x-2y=0$.

$(x_0, y_0)=(1,1)$ is a point of $\mathbb{R}^2$ that is not on an eigenline of $\mathbf{A}$. Let (x_n, y_n) be the iteration sequence generated by $\mathbf{A}$ with initial point $(x_0, y_0)=(1,1)$.

(a) As $k_1<0$, then points of (x_n, y_n) alternate between opposite sides of $l_2: x-2y=0$.

(b) As $\max\{|k_1|, |k_2|\}>1$, then the sequence moves away from $(0,0)$.

(c) As $|-11|>|-2|$, then $k_1=-11$ is the dominant eigenvalue. Hence $\lim\limits_{n\to\infty}\frac{y_n}{x_n}=m$, where $m=-\frac{2}{5}$ is the gradient of the dominant eigenline $l_1: 2x+5y=0$.