MS221: Supplementary Resource Material for Chapter B2, Matrix Transformations

1. *This question has been adapted from the 1999 MS 221 examination paper*

   This question concerns the vectors $\mathbf{a}$ and $\mathbf{b}$, where $\mathbf{a} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

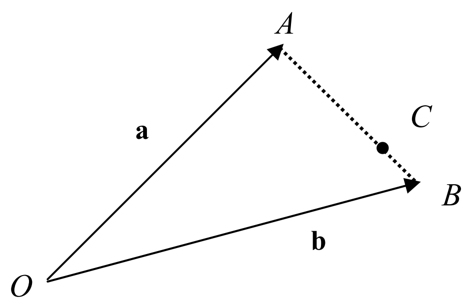
   (a) Find the vector joining the tip of $\mathbf{a}$ to the tip of $\mathbf{b}$.

   (b) Hence find the vector joining the tip of $\mathbf{a}$ to a point one-quarter of the way from the tip of $\mathbf{a}$ to the tip of $\mathbf{b}$.

   (c) Hence, or otherwise, find the vector which joins the origin to the point one-quarter of the way from the tip of $\mathbf{a}$ to the tip of $\mathbf{b}$.

2. *This question has been adapted from the year 2000 MS 221 examination paper*

   In the diagram below, the directed line segments *OA* and *OB* represent the vectors $\mathbf{a}$ and $\mathbf{b}$ respectively, and *C* is a point on *AB* such that $AC = 3CB$



   (a) Find, in terms of $\mathbf{a}$ and $\mathbf{b}$, the vector that represents the directed line segment (i) *AB* (ii) *AC*

   (b) Hence find the vector $\mathbf{c}$, that represents the directed line segment *OC*, in terms of $\mathbf{a}$ and $\mathbf{b}$.

   (c) In the case where $\mathbf{a} = \begin{pmatrix} 4 \\ 8 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 8 \\ 3 \end{pmatrix}$ find the vector $\mathbf{c}$.

3. (a) Determine the matrix that represents an anti-clockwise rotation through $\frac{\pi}{6}$ radians about the origin. Use this matrix to determine the image of the unit square, *S*, under $r_{\frac{\pi}{6}}$ and state its area. Sketch the image, clearly labelling the co-ordinates of the vertices.

   (b) Use two different methods to find the matrix which undoes the effect of $r_{\frac{\pi}{6}}$, and confirm that the image of the unit square that you determined in (a) is mapped by this matrix back onto *S*.

   (c) Give two reasons why the rotations in (a) and (b) are area-preserving.

4. Identify the transformations represented by each of the following matrices and state whether or not they are area preserving. Justify each of your answers by reference to the corresponding determinant.

$$(a)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\quad (b)\begin{pmatrix}4 & 0\\ 0 & 4\end{pmatrix}\quad (c)\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\quad (d)\begin{pmatrix}1 & 5\\ 0 & 1\end{pmatrix}\quad (e)\begin{pmatrix}\frac{\sqrt{3}}{2} & \frac{1}{2}\\ \frac{1}{2} & -\frac{\sqrt{3}}{2}\end{pmatrix}\quad (f)\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$$

5. (a) For the matrices $\mathbf{A}=\begin{pmatrix}2 & 7\\ -3 & 4\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}3 & 11\\ -9 & 5\end{pmatrix}$, show that $\det\mathbf{AB}=\det\mathbf{A}\det\mathbf{B}$.

(b) Prove that $\det\mathbf{AB}=\det\mathbf{A}\det\mathbf{B}$ for all 2×2 matrices $\mathbf{A}$ and $\mathbf{B}$.

6. Let f be the linear transformation that sends $(1,0)$ to $(5,7)$ and $(0,1)$ to $(-5,4)$.

(a) Write down the matrix $\mathbf{A}$ that represents f and calculate its determinant.

(b) What is the area of the triangle with its vertices at $(0,0)$, $(5,7)$ and $(-5,4)$?

7. For each of the following linear transformations, f, write down the matrix $\mathbf{A}$ that represents the transformation:

(a) f scales the plane by a factor of 5 in the direction of the x-axis, and by a factor of 9 in the direction of the y-axis.

(b) f rotates the plane about the origin through $\frac{2\pi}{3}$ in an anticlockwise direction.

(c) f shears the plane parallel to the x-axis in such a way that points at height 1 above the x-axis shift 7 units to the right.

(d) f maps the points $(1,0)$ and $(0,1)$ to the points $(-5,2)$ and $(3,11)$.

8. Let f be the linear transformation represented by the matrix $\begin{pmatrix}7 & 0\\ 11 & -1\end{pmatrix}$. Show by first principles that f is both one-one and onto.

9. Find the single matrix representing each of the following composite transformations, and in each case find the image of the point $(2,-3)$:

(a) A reflection in $y=-x$ followed by an anticlockwise rotation through $\frac{\pi}{4}$.

(b) An anticlockwise rotation through $\frac{\pi}{4}$ followed by a reflection in $y=-x$.

(c) A reflection in $y=x$ followed by a clockwise rotation through $\frac{\pi}{6}$ followed by a second reflection in $y=x$.

10. For each of parts (a), (b) and (c) in question nine, find the single matrix that undoes the combined effect of each of the transformations. Check that your answer is correct by confirming that the image of the point $(2,-3)$ is mapped back onto $(2,-3)$.

11. By determining appropriate matrices, confirm that a reflection in the x-axis followed by a reflection in the y-axis is equivalent to a rotation through π radians about the origin.

12. Determine the matrix that represents the composite transformation $q_{\frac{\pi}{4}} \circ r_{\frac{\pi}{6}}$, and show that its inverse is given by $r_{\frac{\pi}{6}}^{-1} \circ q_{\frac{\pi}{4}}^{-1}$.

13. Let f be the linear transformation represented by the matrix $\begin{pmatrix} 8 & -1 \\ 2 & -\frac{1}{4} \end{pmatrix}$.

Use a property of determinants to show that f represents a flattening, and determine the Cartesian equation of the line which $\mathbb{R}^2$ maps onto.

14. Let f and g be the linear transformations represented by the matrices $\begin{pmatrix} 1 & 5 \\ -7 & 3 \end{pmatrix}$ and $\begin{pmatrix} 2 & -1 \\ 0 & -9 \end{pmatrix}$ respectively.

If the image of a point $P(x, y)$ under the composite transformation fgf is $(2,1)$, determine the position vector of P.

15. Let f be the linear transformation represented by the matrix $\mathbf{A} = \begin{pmatrix} 5 & 1 \\ 6 & -2 \end{pmatrix}$.

(a) Show that f is one-one and onto, and determine f^{-1}.

(b) Find the image of the unit square S under f, and check that is mapped back to S by f^{-1}.

(c) What are the factors by which f and f^{-1} scale areas, and what is the relationship between these factors?

16. Let f be the linear transformation represented by the matrix $\begin{pmatrix} -3 & 5 \\ 1 & 9 \end{pmatrix}$.

(a) Write down the equation of the circle with centre $(2,-3)$ and radius 5.

(b) Find the equation of the image of this circle under f and calculate its area.

17. (a) Write down the affine transformation that maps the points $(0,0)$, $(1,0)$ and $(0,1)$ to the points $(3,7)$, $(2,5)$ and $(-2,9)$ respectively, and state whether or not the orientation is reversed.

(b) Hence write down the area of the triangle with its vertices at $(3,7)$, $(2,5)$ and $(-2,9)$.

18. Find the affine transformation, f, that describes a clockwise rotation about the point $(5,-2)$ through $\frac{\pi}{6}$ radians. Confirm that the centre of rotation is invariant under f.

19. (a) Determine the matrix that represents reflection in the line $y = -x$.

(b) By first translating the point $(0,-3)$ to the origin, find the affine transformation that describes reflection in the line $y = -x - 3$. Check that your answer is correct by finding the image of an arbitrary point on the line $y = -x - 3$.

Answers

1. (a) The vector joining the tip of **a** to the tip of **b** is $\mathbf{b}-\mathbf{a}=\begin{pmatrix}-4\\4\end{pmatrix}$.

(b) Hence the vector joining the tip of **a** to a point one-quarter of the way from the tip of **a** to the tip of **b** is $\frac{1}{4}(\mathbf{b}-\mathbf{a})=\begin{pmatrix}-1\\1\end{pmatrix}$.

(c) Hence the vector which joins the origin to the point one-quarter of the way from the tip of **a** to the tip of **b** is $\mathbf{a}+\frac{1}{4}(\mathbf{b}-\mathbf{a})=\begin{pmatrix}4\\-1\end{pmatrix}$.

2. (a) (i) $\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\mathbf{b}-\mathbf{a}$ (ii) $\overrightarrow{AC}=\frac{3}{4}\overrightarrow{AB}=\frac{3}{4}(\mathbf{b}-\mathbf{a})$

(b) $\mathbf{c}=\overrightarrow{OC}=\overrightarrow{OA}+\overrightarrow{AC}=\mathbf{a}+\frac{3}{4}(\mathbf{b}-\mathbf{a})=\frac{1}{4}(\mathbf{a}+3\mathbf{b})$

(c) $\mathbf{c}=\frac{1}{4}\left[\begin{pmatrix}4\\8\end{pmatrix}+3\begin{pmatrix}8\\3\end{pmatrix}\right]=\begin{pmatrix}7\\\frac{17}{4}\end{pmatrix}$

3. (a) $r_{\frac{\pi}{6}}=\begin{pmatrix}\cos\frac{\pi}{6} & -\sin\frac{\pi}{6}\\ \sin\frac{\pi}{6} & \cos\frac{\pi}{6}\end{pmatrix}=\begin{pmatrix}\frac{\sqrt{3}}{2} & -\frac{1}{2}\\ \frac{1}{2} & \frac{\sqrt{3}}{2}\end{pmatrix}$

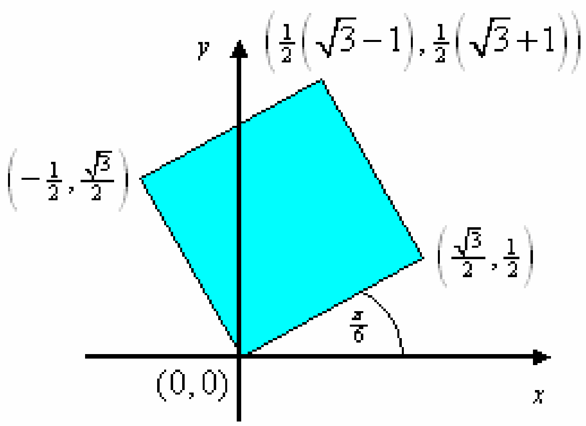
Hence the image of the unit square, S, under $r_{\frac{\pi}{6}}$ is given by

$$\begin{pmatrix}\frac{\sqrt{3}}{2} & -\frac{1}{2}\\ \frac{1}{2} & \frac{\sqrt{3}}{2}\end{pmatrix}\begin{pmatrix}0\\0\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

$$\begin{pmatrix}\frac{\sqrt{3}}{2} & -\frac{1}{2}\\ \frac{1}{2} & \frac{\sqrt{3}}{2}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}\frac{\sqrt{3}}{2}\\\frac{1}{2}\end{pmatrix}$$

$$\begin{pmatrix}\frac{\sqrt{3}}{2} & -\frac{1}{2}\\ \frac{1}{2} & \frac{\sqrt{3}}{2}\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}-\frac{1}{2}\\\frac{\sqrt{3}}{2}\end{pmatrix}$$

$$\begin{pmatrix}\frac{\sqrt{3}}{2} & -\frac{1}{2}\\ \frac{1}{2} & \frac{\sqrt{3}}{2}\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}\frac{1}{2}\left(\sqrt{3}-1\right)\\\frac{1}{2}\left(\sqrt{3}+1\right)\end{pmatrix}$$



The area of the image square will be 1, the same as that of the original square.

(b) ♦ One way of undoing the effect of $r_{\frac{\pi}{6}}$ is to find $r_{-\frac{\pi}{6}}$, which is given by

$$r_{-\frac{\pi}{6}}=\begin{pmatrix}\cos\left(-\frac{\pi}{6}\right) & -\sin\left(-\frac{\pi}{6}\right)\\ \sin\left(-\frac{\pi}{6}\right) & \cos\left(-\frac{\pi}{6}\right)\end{pmatrix}=\begin{pmatrix}\frac{\sqrt{3}}{2} & \frac{1}{2}\\ -\frac{1}{2} & \frac{\sqrt{3}}{2}\end{pmatrix}$$

♦ Alternatively, we can undo the effect of $r_{\frac{\pi}{6}}$ by using the formula for calculating an inverse matrix to find $r_{\frac{\pi}{6}}^{-1}$. As the determinant of all rotation matrices is 1, we have

$$r_{\frac{\pi}{6}}^{-1} = \frac{1}{1}\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$ which is, of course, the same as $r_{-\frac{\pi}{6}}$.

We can now use the inverse matrix to check that the images we calculated in (a) are mapped back onto the vertices of the unit square. We have

$$\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} \frac{1}{2}\left(\sqrt{3}-1\right) \\ \frac{1}{2}\left(\sqrt{3}+1\right) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

(c)
- ♦ All rotations are isometric transformations of the plane and hence preserve area.
- ♦ As the determinant of a rotation matrix is 1, the areas of the images under $r_{\frac{\pi}{6}}$ and $r_{-\frac{\pi}{6}}$ are precisely the same as the areas of the original figures.

4. (a) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ generates a reflection in the line $y = x$. As the determinant of the transformation matrix is 1, this mapping is area preserving.

(b) $\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$ generates a uniform scaling with factor 4. As the determinant of the transformation matrix is $16 \neq 1$, this mapping is not area preserving.

(c) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ generates a rotation about the origin through $\frac{\pi}{2}$ radians anticlockwise. As the determinant of the transformation matrix is 1, this mapping is area preserving.

(d) $\begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix}$ generates an x-shear with factor 5. As the determinant of the transformation matrix is 1, this mapping is area preserving.

(e) $\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}$ is of the form $\begin{pmatrix} \cos\frac{\pi}{6} & \sin\frac{\pi}{6} \\ \sin\frac{\pi}{6} & -\cos\frac{\pi}{6} \end{pmatrix}$, and so represents $q_{\frac{\pi}{12}}$ which is a reflection in the line $y = \tan\left(\frac{\pi}{12}x\right)$. As the determinant of the transformation matrix is 1, this mapping is area preserving.

(f) $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ generates a rotation about the origin through π radians (in either sense). As the determinant of the transformation matrix is 1, this mapping is area preserving.

5. (a) With $\mathbf{A} = \begin{pmatrix} 2 & 7 \\ -3 & 4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 3 & 11 \\ -9 & 5 \end{pmatrix}$, $\det \mathbf{A} = 29$, $\det \mathbf{B} = 114$ and $\det \mathbf{A} \det \mathbf{B} = 3306$.

Also $\mathbf{AB} = \begin{pmatrix} 2 & 7 \\ -3 & 4 \end{pmatrix}\begin{pmatrix} 3 & 11 \\ -9 & 5 \end{pmatrix} = \begin{pmatrix} -57 & 57 \\ -45 & -13 \end{pmatrix}$ and so $\det \mathbf{AB} = 3306$.

Hence for these particular matrices $\mathbf{A}$ and $\mathbf{B}$, we have $\det \mathbf{AB} = \det \mathbf{A} \det \mathbf{B}$.

(b) To prove that $\det \mathbf{AB} = \det \mathbf{A} \det \mathbf{B}$ for *all* 2×2 matrices $\mathbf{A}$ and $\mathbf{B}$ we must consider the problem in general terms.

Let $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$

Then $\det \mathbf{A} \det \mathbf{B} = (a_{11}a_{22} - a_{21}a_{12})(b_{11}b_{22} - b_{21}b_{12})$

Also $\mathbf{AB} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}$

$$\begin{aligned} \det \mathbf{AB} &= (a_{11}b_{11} + a_{12}b_{21})(a_{21}b_{12} + a_{22}b_{22}) - (a_{21}b_{11} + a_{22}b_{21})(a_{11}b_{12} + a_{12}b_{22}) \\ &= a_{11}b_{11}a_{21}b_{12} + a_{11}b_{11}a_{22}b_{22} + a_{12}b_{21}a_{21}b_{12} + a_{12}b_{21}a_{22}b_{22} \\ &\quad - a_{21}b_{11}a_{11}b_{12} - a_{21}b_{11}a_{12}b_{22} - a_{22}b_{21}a_{11}b_{12} - a_{22}b_{21}a_{12}b_{22} \\ &= a_{11}b_{11}a_{22}b_{22} + a_{12}b_{21}a_{21}b_{12} - a_{21}b_{11}a_{12}b_{22} - a_{22}b_{21}a_{11}b_{12} \\ \therefore \quad &= [a_{11}b_{11}a_{22}b_{22} - a_{22}b_{21}a_{11}b_{12}] + [a_{12}b_{21}a_{21}b_{12} - a_{21}b_{11}a_{12}b_{22}] \\ &= a_{11}a_{22}(b_{11}b_{22} - b_{21}b_{12}) - a_{12}a_{21}(b_{11}b_{22} - b_{21}b_{12}) \\ &= (a_{11}a_{22} - a_{12}a_{21})(b_{11}b_{22} - b_{21}b_{12}) \\ &= \det \mathbf{A} \det \mathbf{B} \end{aligned}$$

6. (a) As f is such that $(1,0) \mapsto (5,7)$ and $(0,1) \mapsto (-5,4)$, then it is represented by the matrix $A = \begin{pmatrix} 5 & -5 \\ 7 & 4 \end{pmatrix}$. The determinant of $\mathbf{A}$ is $20 - (-35) = 55$.

(b) As $\det \mathbf{A} = 55$, the area of the new triangle will be 55 times larger than the area of the original triangle, which is $\frac{1}{2}$. Hence the area of the new triangle is $55 \times \frac{1}{2} = 27.5$.

7. (a) $\mathbf{A} = \begin{pmatrix} 5 & 0 \\ 0 & 9 \end{pmatrix}$ (b) $\mathbf{A} = \begin{pmatrix} \cos\frac{2\pi}{3} & -\sin\frac{2\pi}{3} \\ \sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$ (c) $\mathbf{A} = \begin{pmatrix} 1 & 7 \\ 0 & 1 \end{pmatrix}$ (d) $\mathbf{A} = \begin{pmatrix} -5 & 3 \\ 2 & 11 \end{pmatrix}$

8. To show that f is one-one, we start by supposing that (r,s) and (u,v) are points such that $f(r,s) = \mathrm{f}(u,v)$. Then

$\begin{pmatrix} 7 & 0 \\ 11 & -1 \end{pmatrix}\begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} 7 & 0 \\ 11 & -1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$, from which $\begin{pmatrix} 7r \\ 11r - s \end{pmatrix} = \begin{pmatrix} 7u \\ 11u - v \end{pmatrix}$

Equating components gives $7r = 7u$ and $11r - s = 11u - v$. Hence $r = u$ and consequently $s = v$. It follows that $(r,s) = (u,v)$ and hence that f is one-one.

To show that f is onto we let (u,v) be an arbitrary point in the codomain, $\mathbb{R}^2$.

Now for (u,v) to be the image of a point (x,y) in the domain, we require $f(x,y) = (u,v)$.

i.e. $\begin{pmatrix} 7 & 0 \\ 11 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$, or equivalently $\begin{pmatrix} 7x \\ 11x - y \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$

Hence the component equations are $7x = u$ and $11x - y = v$.

The first of these gives $x = \frac{1}{7}u$ and substitution into the second gives $y = \frac{11}{7}u - v$.

Hence one point that maps to (u,v) is $(x,y) = (\frac{1}{7}u, \frac{11}{7}u - v)$. Since (u,v) is an arbitrary point in the codomain, $\mathbb{R}^2$, we conclude that $f\left(\mathbb{R}^2\right) = \mathbb{R}^2$. Hence f is onto.

9. (a) The matrix representing reflection in $y = -x$ is $\mathbf{A} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ and the matrix representing an anticlockwise rotation through $\frac{\pi}{4}$ is $\mathbf{B} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$.

Hence the matrix representing the composite transformation "a reflection in $y = -x$ followed by an anticlockwise rotation through $\frac{\pi}{4}$" is $\mathbf{BA} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$.

The image of $(2,-3)$ under the composite transformation is found by taking the matrix product $\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} \frac{5}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$. Hence the image of $(2,-3)$ is $\left(\frac{5}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

(b) Here we have been asked to perform the same two transformations as in (a), but in the reverse order. Hence the matrix representing the composite transformation "an anticlockwise rotation through $\frac{\pi}{4}$ followed by a reflection in $y = -x$" is

$$\mathbf{AB} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

The image of $(2,-3)$ under this composite transformation is $\begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{5}{\sqrt{2}} \end{pmatrix}$, which corresponds to the point with co-ordinates $\left(\frac{1}{\sqrt{2}}, -\frac{5}{\sqrt{2}}\right)$.

(c) The matrix representing reflection in the line $y = x$ is $\mathbf{C} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and the matrix representing a clockwise rotation through $\frac{\pi}{6}$ is $\mathbf{D} = \begin{pmatrix} \cos\left(-\frac{\pi}{6}\right) & -\sin\left(-\frac{\pi}{6}\right) \\ \sin\left(-\frac{\pi}{6}\right) & \cos\left(-\frac{\pi}{6}\right) \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$.

Hence the composite transformation "a reflection in $y = x$ followed by a clockwise rotation through $\frac{\pi}{6}$ followed by a second reflection in $y = x$" is given by

$$\mathbf{CDC} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

The image of $(2,-3)$ under this composite transformation is $\begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} \sqrt{3}+\frac{3}{2} \\ 1-\frac{3\sqrt{3}}{2} \end{pmatrix}$,

which corresponds to the point with co-ordinates $\left(\sqrt{3}+\frac{3}{2}, 1-\frac{3\sqrt{3}}{2}\right)$.

10. In each case it is sufficient to calculate the inverse matrix of the composite transformation, and then show that this inverse maps the image of $(2,-3)$ back onto $(2,-3)$. We have

(a) The matrix $\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$ is actually self-inverse, and so $\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$

Hence $\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}^{-1}\begin{pmatrix} \frac{5}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \frac{5}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$, as required.

(b) The matrix $\begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ is also self-inverse, and so $\begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}^{-1} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

Hence $\begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}^{-1}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{5}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{5}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$, as required.

(c) The inverse of the matrix $\begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$ is $\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$

Hence $\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} \sqrt{3}+\frac{3}{2} \\ 1-\frac{3\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$, as required.

11. The matrices representing reflection in the x and y axes are $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ respectively.

Hence the matrix representing reflection in the x-axis followed by reflection in the y-axis is $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, which is precisely the matrix representing a rotation through π radians about the origin.

12. With $q_{\frac{\pi}{4}} = \begin{pmatrix} \cos\left(\frac{\pi}{2}\right) & \sin\left(\frac{\pi}{2}\right) \\ \sin\left(\frac{\pi}{2}\right) & -\cos\left(\frac{\pi}{2}\right) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\mathrm{r}_{\frac{\pi}{6}} = \begin{pmatrix} \cos\left(\frac{\pi}{6}\right) & -\sin\left(\frac{\pi}{6}\right) \\ \sin\left(\frac{\pi}{6}\right) & \cos\left(\frac{\pi}{6}\right) \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$

We have $q_{\frac{\pi}{4}} \circ \mathrm{r}_{\frac{\pi}{6}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$

This is another self-inverse matrix, and so $\left(q_{\frac{\pi}{4}} \circ \mathrm{r}_{\frac{\pi}{6}}\right)^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$

Also $\mathrm{r}_{\frac{\pi}{6}}^{-1} = \mathrm{r}_{-\frac{\pi}{6}} = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$ and $q_{\frac{\pi}{4}}^{-1} = q_{\frac{\pi}{4}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\therefore \mathrm{r}_{\frac{\pi}{6}}^{-1} \circ q_{\frac{\pi}{4}}^{-1} = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} = \left(q_{\frac{\pi}{4}} \circ \mathrm{r}_{\frac{\pi}{6}}\right)^{-1}$

13. $\begin{vmatrix} 8 & -1 \\ 2 & -\frac{1}{4} \end{vmatrix} = 0$, and so f represents a flattening.

As $\begin{pmatrix} 8 & -1 \\ 2 & -\frac{1}{4} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 8 & -1 \\ 2 & -\frac{1}{4} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -\frac{1}{4} \end{pmatrix} = -\frac{1}{4}\begin{pmatrix} 4 \\ 1 \end{pmatrix}$

Then both the images $f(1,0)$ and $f(0,1)$ lie on the line $x - 4y = 0$.

Hence all of $\mathbb{R}^2$ collapses onto the line $x - 4y = 0$.

14. $fgf = \begin{pmatrix} 1 & 5 \\ -7 & 3 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 0 & -9 \end{pmatrix}\begin{pmatrix} 1 & 5 \\ -7 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 5 \\ -7 & 3 \end{pmatrix}\begin{pmatrix} 9 & 7 \\ 63 & -27 \end{pmatrix} = \begin{pmatrix} 324 & -128 \\ 126 & -130 \end{pmatrix}$

So if $fgf\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, then $\begin{pmatrix} 324 & -128 \\ 126 & -130 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Then $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 324 & -128 \\ 126 & -130 \end{pmatrix}^{-1}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{65}{12996} & -\frac{16}{3249} \\ \frac{7}{1444} & -\frac{9}{722} \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{11}{2166} \\ -\frac{1}{361} \end{pmatrix}$

15. (a) As $\det \mathbf{A} = -16 \neq 0$, it follows from the boxed result on page 42 that f is invertible and hence both one-one and onto.

The inverse mapping, f^{-1}, is the linear transformation represented by $\mathbf{A}^{-1} = \begin{pmatrix} \frac{1}{8} & \frac{1}{16} \\ \frac{3}{8} & -\frac{5}{16} \end{pmatrix}$.

(b) The image, $f(S)$, of the unit square has its vertices at the points with position vectors

$\begin{pmatrix} 5 & 1 \\ 6 & -2 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$; $\begin{pmatrix} 5 & 1 \\ 6 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$; $\begin{pmatrix} 5 & 1 \\ 6 & -2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$; $\begin{pmatrix} 5 & 1 \\ 6 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$

i.e $f(S)$ is the parallelogram with vertices $(0,0)$, $(5,6)$, $(1,-2)$ and $(6,4)$

Under f^{-1}, the image $f(S)$ maps back to the figure with vertices given by

$\begin{pmatrix} \frac{1}{8} & \frac{1}{16} \\ \frac{3}{8} & -\frac{5}{16} \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$; $\begin{pmatrix} \frac{1}{8} & \frac{1}{16} \\ \frac{3}{8} & -\frac{5}{16} \end{pmatrix}\begin{pmatrix} 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$; $\begin{pmatrix} \frac{1}{8} & \frac{1}{16} \\ \frac{3}{8} & -\frac{5}{16} \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$; $\begin{pmatrix} \frac{1}{8} & \frac{1}{16} \\ \frac{3}{8} & -\frac{5}{16} \end{pmatrix}\begin{pmatrix} 6 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Which are the vertices of the unit square, .

(c) Under f, areas are scaled by $|\det \mathbf{A}| = |-16| = 16$

Under f^{-1}, areas are scaled by $|\det \mathbf{A}^{-1}| = |-\frac{1}{16}| = \frac{1}{16}$

Each area scale factor is the reciprocal of the other.

16. (a) $(x-2)^2 + (y+3)^2 = 25$

(b) Let (X, Y) be the image under f of a point (x, y) on the circle $(x-2)^2 + (y+3)^2 = 25$. Then

$$\begin{pmatrix} -3 & 5 \\ 1 & 9 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix} \text{ from which } \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 & 5 \\ 1 & 9 \end{pmatrix}^{-1}\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} -\frac{9}{32} & \frac{5}{32} \\ \frac{1}{32} & \frac{3}{32} \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} -\frac{9}{32}X + \frac{5}{32}Y \\ \frac{1}{32}X + \frac{3}{32}Y \end{pmatrix}$$

Hence $x = -\frac{9}{32}X + \frac{5}{32}Y$ and $y = \frac{1}{32}X + \frac{3}{32}Y$.

Substituting for x and y into the equation $(x-2)^2 + (y+3)^2 = 25$ of the original circle gives

$$\left(-\tfrac{9}{32}X + \tfrac{5}{32}Y - 2\right)^2 + \left(\tfrac{1}{32}X + \tfrac{3}{32}Y + 3\right)^2 = 25$$

And if we replace X and Y by x and y and expand the brackets, we arrive at

$$\frac{41}{512}x^2 + \frac{21}{16}x + \frac{17}{512}y^2 - \frac{1}{16}y - \frac{21}{256}xy - 12 = 0$$

Or equivalently, $41x^2 + 672x + 17y^2 - 32y - 42xy - 6144 = 0$ (which is the equation of an ellipse in non-standard position).

The area of the original circle was $\pi \times 5^2 = 25\pi$.

As the determinant of the transformation matrix $\begin{pmatrix} -3 & 5 \\ 1 & 9 \end{pmatrix}$ is -32, the area of the image of the given circle is $|-32| \times 25\pi = 800\pi$.

17. Here we make use of the two boxed results on page 49 of B2.

(a) The affine transformation f has the rule $\mathbf{x} \mapsto \mathbf{Ax} + \mathbf{a}$ where $\mathbf{A} = \begin{pmatrix} -1 & -5 \\ -2 & 2 \end{pmatrix}$ and $\mathbf{a} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$.

As $\det \mathbf{A} = -12 < 0$, the orientation is reversed.

(b) Since the translation through the vector $\mathbf{a}$ has no effect on areas, it follows that the affine transformation scales areas by the factor $|\det \mathbf{A}| = 12$.

As the area of the triangle with its vertices at $(0,0)$, $(1,0)$ and $(0,1)$ is $\frac{1}{2}$, it follows that the area of the image triangle is $12 \times \frac{1}{2} = 6$.

18. Firstly, we map the centre of rotation to the origin by the inverse translation $t_{-5,2}$.

Hence $\mathbf{x} \mapsto \mathbf{x} + \begin{pmatrix} -5 \\ 2 \end{pmatrix}$

Now we use $r_{\frac{\pi}{6}} = \begin{pmatrix} \cos\frac{\pi}{6} & -\sin\frac{\pi}{6} \\ \sin\frac{\pi}{6} & \cos\frac{\pi}{6} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$ to perform the rotation.

Hence our original point $\mathbf{x}$ becomes

$$\begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}\left(\mathbf{x} + \begin{pmatrix} -5 \\ 2 \end{pmatrix}\right) = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}\mathbf{x} + \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} -5 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}\mathbf{x} + \begin{pmatrix} -\frac{5}{2}\sqrt{3} - 1 \\ \sqrt{3} - \frac{5}{2} \end{pmatrix}$$

Finally, we return the transformed plane back to its original position by applying the transformation $t_{5,-2}$. Hence the complete affine transformation is given by

$$f(\mathbf{x}) = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}\mathbf{x} + \begin{pmatrix} -\frac{5}{2}\sqrt{3} - 1 \\ \sqrt{3} - \frac{5}{2} \end{pmatrix} + \begin{pmatrix} 5 \\ -2 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}\mathbf{x} + \begin{pmatrix} 4 - \frac{5}{2}\sqrt{3} \\ \sqrt{3} - \frac{9}{2} \end{pmatrix}$$

We must now check that the centre of rotation, $(5,-2)$, is invariant under f. We have

$$f\begin{pmatrix} 5 \\ -2 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} 5 \\ -2 \end{pmatrix} + \begin{pmatrix} 4 - \frac{5}{2}\sqrt{3} \\ \sqrt{3} - \frac{9}{2} \end{pmatrix} = \begin{pmatrix} \frac{5}{2}\sqrt{3} + 1 \\ \frac{5}{2} - \sqrt{3} \end{pmatrix} + \begin{pmatrix} 4 - \frac{5}{2}\sqrt{3} \\ \sqrt{3} - \frac{9}{2} \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}, \text{ as required.}$$

19. (a) The matrix corresponding to reflection in the line $y = -x$ is $q_{\frac{3\pi}{4}} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$.

(b) Firstly we translate the plane 3 units up, so that the line $y = -x - 3$ maps onto the line $y = -x$.

Hence $\mathbf{x} \mapsto \mathbf{x} + \begin{pmatrix} 0 \\ 3 \end{pmatrix}$. Now we use $q_{\frac{3\pi}{4}}$ to reflect $\mathbf{x} + \begin{pmatrix} 0 \\ 3 \end{pmatrix}$ in the translated line. This gives

$$q_{\frac{3\pi}{4}}\left(\mathbf{x} + \begin{pmatrix} 0 \\ 3 \end{pmatrix}\right) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\mathbf{x} + \begin{pmatrix} -3 \\ 0 \end{pmatrix}$$

Finally, we return the line of reflection to its original position by applying the transformation $t_{0,-3}$. Hence the complete affine transformation is given by

$$\mathrm{f}(\mathbf{x}) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\mathbf{x} + \begin{pmatrix} -3 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -3 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\mathbf{x} + \begin{pmatrix} -3 \\ -3 \end{pmatrix}$$

We will now check the rule for this affine transformation f by finding the image of an arbitrary point on the line $y = -x - 3$. We have

$$\mathrm{f}\begin{pmatrix} x \\ -x-3 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ -x-3 \end{pmatrix} + \begin{pmatrix} -3 \\ -3 \end{pmatrix} = \begin{pmatrix} x+3 \\ -x \end{pmatrix} + \begin{pmatrix} -3 \\ -3 \end{pmatrix} = \begin{pmatrix} x \\ -x-3 \end{pmatrix}$$

Which confirms that every point on the line $y = -x - 3$ is mapped to itself under f .