MS221: Supplementary Resource Material for Chapter B1, Iteration

1. (a) Determine the first 5 terms of the iteration sequence $x_{n+1} = -\frac{1}{12}x_n^2 + \frac{17}{16}x_n + 4$ for each of the following initial terms, x_0, giving your answers correct to three significant figures:

   (i) $x_0 = -10$ (ii) $x_0 = -5$ (iii) $x_0 = 1$ (iv) $x_0 = 12$

   (b) Taking $[-15, 20]$ as the domain, plot the graphs of $f(x) = -\frac{1}{12}x^2 + \frac{17}{16}x + 4$ and $g(x) = x$ on the same set of axes. Use staircase diagrams to illustrate each of your answers to part (a).

   (c) Determine the exact values of the fixed points of $f(x)$, giving your answers in surdform.

   (d) Use your previous answers to predict the long term behaviour of the iteration sequences given by $x_{n+1} = -\frac{1}{12}x_n^2 + \frac{17}{16}x_n + 4$ in each of the cases

   (i) $x_0 = -10$ (ii) $x_0 = -5$ (iii) $x_0 = 1$ (iv) $x_0 = 12$.

   (e) Find $f'(x)$ and hence write down the gradient of $f(x)$ at each of the fixed points that you identified in (c).

   (f) Hence classify each of the fixed points as attracting, repelling, indifferent or super-attracting.

   (g) Are your predictions of the long term behaviour of the iteration sequences in (d) consistent with your classifications of the nature of the fixed points in (f)?

   (h) Use the gradient criterion to find an interval of attraction for an attracting fixed point of the function $f(x)$.

2. Let f, g and h be the functions

   (i) $f(x) = \ln x \quad , \; x \in (0, \infty)$

   (ii) $g(x) = \tan(x) \quad , \; x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

   (iii) $h(x) = x^2 + 1 \quad , \; x \in \mathbb{R}$

   (a) Draw up a table showing clearly all nine potential combinations of composite functions.

   (b) For each of the possible combinations of composite functions that you identified in (b), either

   (i) Give the definition of the composite function using two line notation, or

   (ii) Give a reason why the composite function cannot be formed..

3. For the function $f(x) = x^2 - \frac{23}{24}$

(a) Find the fixed points of $f(x)$, giving your answers in exact surdform, and classify them as attracting, repelling, indifferent or super-attracting.

On page 34 of Chapter B1 we saw that

Distinct numbers a and b in the domain of a real function f form a 2-cycle of $f(x)$ if $f(a) = b$ and $f(b) = a$.

(b) Use this definition to formulate and solve a pair of simultaneous equations to identify any 2-cycles of $f(x)$. Hint: You may find this factorisation helpful

$$576a^4 - 1104a^2 - 576a - 23 = \left(24a^2 - 24a - 23\right)\left(24a^2 + 24a + 1\right)$$

(c) For the values of *a* and *b* that you found above, prove that the two cycle is attracting by showing that $\left| f'(a) f'(b) \right| < 1$.

(d) Take $x_0 = 1$ and calculate sufficient iterates to convince yourself that the iteration sequence $x_{n+1} = f(x_n)$ generates the 2-cycle identified above.

The course text also states that

The solutions of the 2-cycle equation $ff(x) = x$ are either fixed points of f or members of 2-cycles of f.

(e) Use this definition to confirm that the 2-cycle you determined above is correctly specified by this alternative method of calculation, and that the values of *a* and *b* are either fixed points of f or members of 2-cycles of f.

4. (a) Expand $(3p - 2q)^6$ using the binomial expansion.

(b) Determine the coefficient of x^5 in the binomial expansion of $(1 - 3x)^{17}$.

(c) Find the constant term in the expansion of $\left(x - \frac{1}{x}\right)^{16}$.

5. (a) A committee of five is to be elected from a candidate list of eighteen. How many possible committees can be formed?

(b) If a particular husband and wife have to be chosen as secretary and treasurer respectively, in how many ways can the committee be formed? What is the significance of the word respectively?

(c) A committee of 4 men and 4 ladies is to be elected from a candidate list of 7 men and 15 ladies. In how many ways can this be done?

(d) If a particular husband and wife have to be chosen as secretary and treasurer respectively, in how many ways can the committee be formed?

(e) There are 18 horses in a race. In how many ways can the first five winning positions be filled?

4. (a) $(3p-2q)^6$

$= 729p^6 - 2916p^5q + 4860p^4q^2 - 4320p^3q^3 + 2160p^2q^4 - 576pq^5 + 64q^6$

(b) $\left[x^5\right] = {}^{17}C_5 \cdot (1)^{12} \cdot (-3)^5 = -1503684$

(c) The constant term in the expansion of $\left(x-\frac{1}{x}\right)^{16}$ is ${}^{16}C_8 \cdot x^8 \cdot \left(-\frac{1}{x}\right)^8 = 12870$.

5. (a) As the members of a committee are not ordered, this is a combinations problem rather than a permutations one. Hence the number of possible committees is ${}^{18}C_5 = 8568$.

(b)
- ♦ If we remove the husband and wife, we must then select three people from the remaining 16. Hence the number of committees is now ${}^{16}C_3 = 560$.
- ♦ If the word 'respectively' was omitted, there would be two choices for secretary and treasurer and consequently $2 \times {}^{16}C_3 = 1120$ different committees could be formed.

(c) There are ${}^{7}C_4 = 35$ ways of making 4 unordered selections from 7 men

And ${}^{15}C_4 = 1365$ ways of making 4 unordered selections from 15 ladies

Hence there are $35 \times 1365 = 47775$ ways of making this committee.

(d) We now have to select 3 men from the remaining 6 and 3 ladies from the remaining 14. Hence the number of committees is ${}^{6}C_3 \times {}^{14}C_3 = 20 \times 364 = 7280$

(e) As the finishing order in a horse race is important, we have to determine how many ordered selections of 5 horses can be made from 18. This is a permutations problem and is given by ${}^{18}P_5 = 1028160$.

Answers:

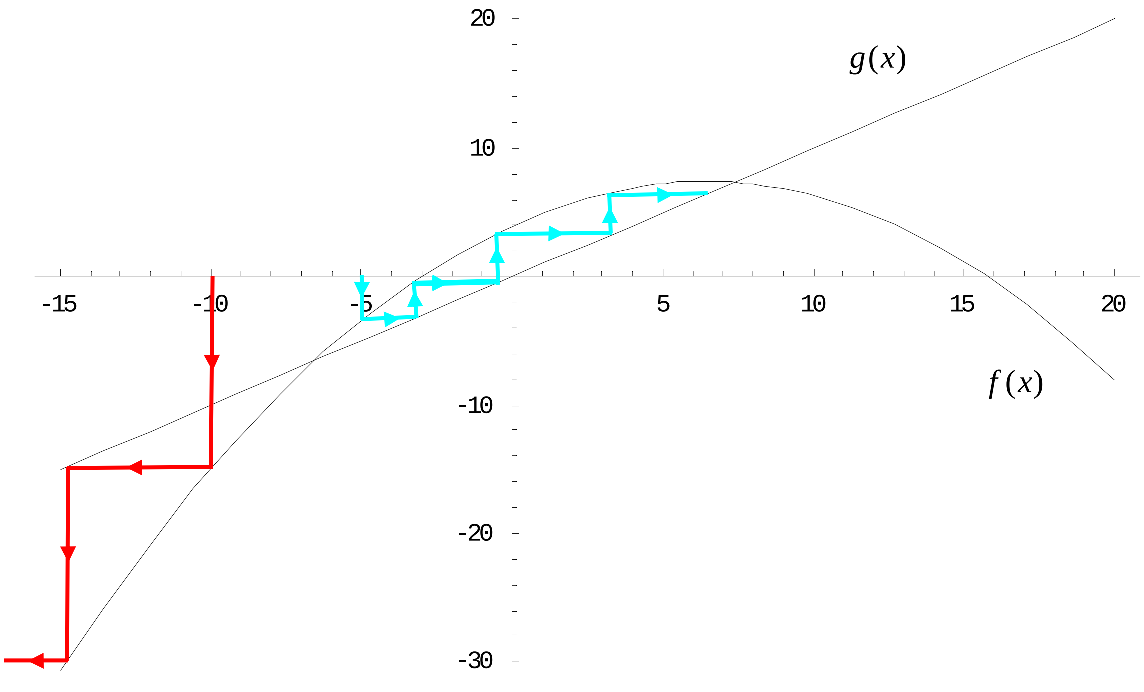
1. (a) (*i*) $x_0 = -10.0,\ x_1 = -15.0,\ x_2 = -30.5,\ x_3 = -106,\ x_4 = -1050$

(*ii*) $x_0 = -5.00,\ x_1 = -3.40,\ x_2 = -0.569,\ x_3 = 3.37,\ x_4 = 6.63$

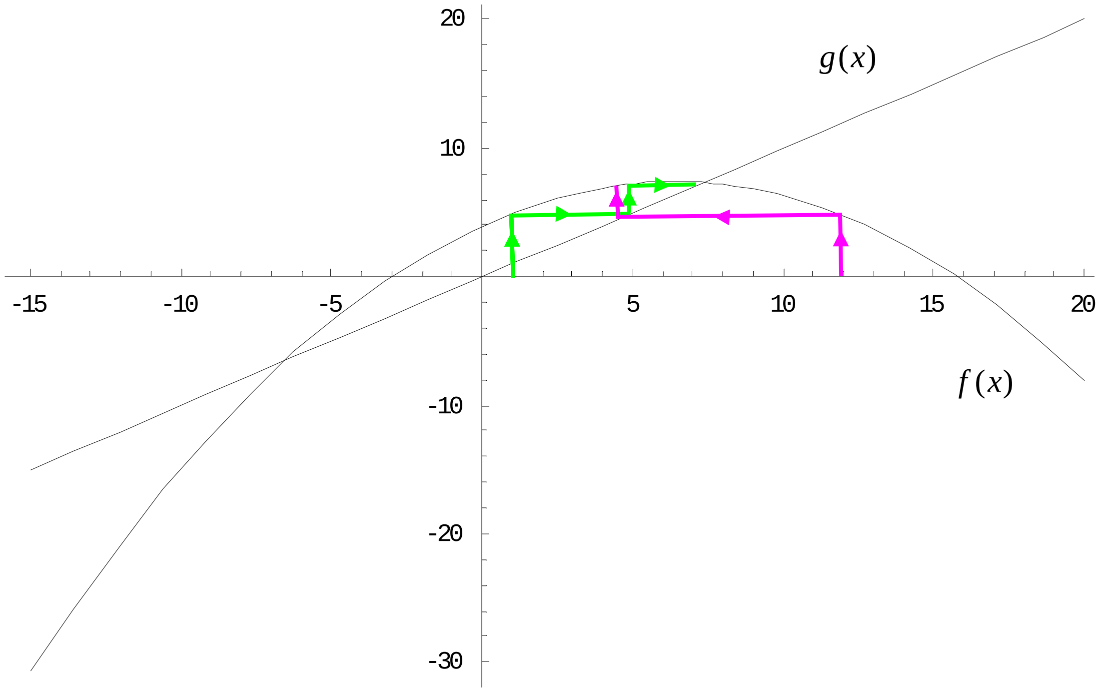
(*iii*) $x_0 = 1.00,\ x_1 = 4.98,\ x_2 = 7.22,\ x_3 = 7.33,\ x_4 = 7.31$

(*iv*) $x_0 = 12.0,\ x_1 = 4.75,\ x_2 = 7.17,\ x_3 = 7.33,\ x_4 = 7.31$

(b) The iteration sequences (i) and (ii) are shown in red and cyan respectively.



The iteration sequences (iii) and (iv) are shown in green and magenta respectively.



$$\left(a^2-\frac{23}{24}\right)^2-\frac{23}{24}=a$$

$$\therefore a^4-\frac{23}{12}a^2+\frac{529}{576}-\frac{23}{24}-a=0$$

$$\therefore a^4-\frac{23}{12}a^2-a-\frac{23}{576}=0$$

$$\therefore 576a^4-1104a^2-576a-23=0$$

$$\therefore \left(24a^2-24a-23\right)\left(24a^2+24a+1\right)=0 \qquad \text{[By the hint]}$$

Now the first of these two factors is just the fixed point equation for $f(x)$. Hence its zeros are $a=\frac{1}{12}\left(6\pm\sqrt{174}\right)$, which are the fixed points of $f(x)$.

The second equation solves to give $a=\frac{1}{12}\left(-6\pm\sqrt{30}\right)$.

When $a=\dfrac{1}{12}\left(-6-\sqrt{30}\right)$, $b=\left[\dfrac{1}{12}\left(-6-\sqrt{30}\right)\right]^2-\dfrac{23}{24}=\dfrac{1}{12}\left(-6+\sqrt{30}\right)$

And when $a=\dfrac{1}{12}\left(-6+\sqrt{30}\right)$, $b=\left[\dfrac{1}{12}\left(-6+\sqrt{30}\right)\right]^2-\dfrac{23}{24}=\dfrac{1}{12}\left(-6-\sqrt{30}\right)$

Hence, by symmetry, there is one 2-cycle of $f(x)$, and this has values $x=\frac{1}{12}\left(-6\pm\sqrt{30}\right)$.

(c) We have already seen that $f'(x)=2x$. Hence, for $x=\frac{1}{12}\left(-6\pm\sqrt{30}\right)$, we have

$$\left|f'(a)f'(b)\right|=\left|\left[2\times\tfrac{1}{12}\left(-6+\sqrt{30}\right)\right]\times\left[2\times\tfrac{1}{12}\left(-6-\sqrt{30}\right)\right]\right|=\frac{1}{6}<1$$

Hence, as $\left|f'(a)f'(b)\right|<1$, the 2-cycle is attracting.

(d) The key sequence (valid for most models of calculator)

1 ENTER 2nd ANS x^2 - 23 ÷ 24 ENTER ENTER ENTER …

shows that the sequence of iterates very quickly settles into the attracting two cycle and oscillates between $x=\frac{1}{12}\left(-6\pm\sqrt{30}\right)$.

(e) $ff(x)=x \Rightarrow \left(x^2-\frac{23}{24}\right)^2-\frac{23}{24}=x$.

This is precisely the same equation that we solved for a in part (b) and so it reduces to $\left(24x^2-24x-23\right)\left(24x^2+24x+1\right)=0$.

Hence either $24x^2-24x-23=0$ whose roots give the fixed points of $f(x)$, or $24x^2+24x+1=0$ whose roots form the 2-cycle identified in (b).

$h \circ f : (0, \infty) \to \mathbb{R}$

$x \mapsto (\ln x)^2 + 1$

- $h \circ g$ can be formed because the image set of g, $\mathbb{R}$, is a subset of the domain of h, $\mathbb{R}$. The two line notation for the composite function $h \circ g$ is

  $h \circ g : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$

  $x \mapsto \tan^2 x + 1$

- $h \circ h$ can be formed because the image set of h, $[1, \infty)$, is a subset of the domain of h, $\mathbb{R}$. The two line notation for the composite function $h \circ h$ is

  $h \circ h : \mathbb{R} \to \mathbb{R}$

  $x \mapsto \left(x^2 + 1\right)^2 + 1$

3 (a) The fixed points of $f(x)$ are the solutions to the equation $f(x) = x$. That is, the solutions to $x^2 - \frac{23}{24} = x$ or equivalently to $24x^2 - 24x - 23 = 0$

$$\therefore x = \frac{24 \pm \sqrt{576 - 4 \times 24 \times (-23)}}{48} = \frac{24 \pm \sqrt{2784}}{48} = \frac{24 \pm 4\sqrt{174}}{48} = \frac{1}{12}\left(6 \pm \sqrt{174}\right)$$

So the fixed points of $f(x)$ are at $x = \frac{1}{12}\left(6 - \sqrt{174}\right)$ and $x = \frac{1}{12}\left(6 + \sqrt{174}\right)$

Now with $f(x) = x^2 - \frac{23}{24}$ we have $f'(x) = 2x$

$$\therefore f'\left(\tfrac{1}{12}\left[6 - \sqrt{174}\right]\right) = 2 \times \tfrac{1}{12}\left[6 - \sqrt{174}\right] \approx -1.2$$

Hence $\left| f'\left(\frac{1}{12}\left[6 - \sqrt{174}\right]\right)\right| > 1$, and so the fixed point at $x = \frac{1}{12}\left(6 - \sqrt{174}\right)$ is repelling.

Similarly, $f'\left(\frac{1}{12}\left[6 + \sqrt{174}\right]\right) = 2 \times \frac{1}{12}\left[6 + \sqrt{174}\right] \approx 3.2$

Hence $\left| f'\left(\frac{1}{12}\left[6 + \sqrt{174}\right]\right)\right| > 1$, and so the fixed point at $x = \frac{1}{12}\left(6 - \sqrt{174}\right)$ is also repelling.

$\therefore$ Both fixed points of $f(x)$ are repelling.

(b) $f(a) = b \Rightarrow a^2 - \frac{23}{24} = b$ and similarly $f(b) = a \Rightarrow b^2 - \frac{23}{24} = a$

Substituting for *b* from the first of these equations into the second gives

$$x = \tfrac{1}{8}\left(3+\sqrt{3081}\right) - \left[\tfrac{99}{8} - \tfrac{1}{8}\left(3+\sqrt{3081}\right)\right] = \tfrac{1}{8}\left(2\sqrt{3081}-93\right)$$

Hence an interval of attraction for $x = \tfrac{1}{8}\left(3+\sqrt{3081}\right)$ is $\left(\tfrac{1}{8}\left(2\sqrt{3081}-93\right), \tfrac{99}{8}\right)$.

2. (a)

		First Function		
Second Function	$\circ$	f	g	h
	f	$f \circ f$	$f \circ g$	$f \circ h$
	g	$g \circ f$	$g \circ g$	$g \circ h$
	h	$h \circ f$	$h \circ g$	$h \circ h$

(b) $f(x) = \ln x$ has domain $(0, \infty)$, image set $\mathbb{R}$ and codomain $\mathbb{R}$.

$g(x) = \tan x$ has domain $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, image set $\mathbb{R}$ and codomain $\mathbb{R}$.

$h(x) = x^2 + 1$ has domain $\mathbb{R}$, image set $[1, \infty)$ and codomain $\mathbb{R}$.

Hence

- $f \circ f$ cannot be formed because the image set of f, $\mathbb{R}$, is not a subset of the domain of f, $(0, \infty)$.
- $f \circ g$ cannot be formed because the image set of g, $\mathbb{R}$, is not a subset of the domain of f, $(0, \infty)$.
- $f \circ h$ can be formed because the image set of h, $[1, \infty)$, is a subset of the domain of f, $(0, \infty)$. The two line notation for the composite function $f \circ h$ is

$$f \circ h : \mathbb{R} \to \mathbb{R}$$
$$x \mapsto \ln\left(x^2 + 1\right)$$

- $g \circ f$ cannot be formed because the image set of f, $\mathbb{R}$, is not a subset of the domain of g, $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.
- $g \circ g$ cannot be formed because the image set of g, $\mathbb{R}$, is not a subset of the domain of g, $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.
- $g \circ h$ cannot be formed because the image set of h, $[1, \infty)$, is not a subset of the domain of g, $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.
- $h \circ f$ can be formed because the image set of f, $\mathbb{R}$, is a subset of the domain of h, $\mathbb{R}$. The two line notation for the composite function $h \circ f$ is

(c) The fixed points of $f(x)$ are the solutions to the equation $f(x) = x$. Hence $-\frac{1}{12}x^2 + \frac{17}{16}x + 4 = x$, from which $48x = -4x^2 + 51x + 192$ and $4x^2 - 3x - 192 = 0$

$$\therefore x = \frac{3 \pm \sqrt{9 - 4 \times 4 \times (-192)}}{8} = \frac{3 \pm \sqrt{3081}}{8} = \frac{1}{8}\left(3 \pm \sqrt{3081}\right)$$

Hence the fixed points of $f(x)$ are $\frac{1}{8}\left(3 - \sqrt{3081}\right)$ and $\frac{1}{8}\left(3 + \sqrt{3081}\right)$

(d) With $x_0 = -10$ the sequence of iterates diverges, but with the other three starting values it converges to the larger of the two fixed points at $x = \frac{1}{8}\left(3 + \sqrt{3081}\right)$.

(e) With $f(x) = -\frac{1}{12}x^2 + \frac{17}{16}x + 4$ we have $f'(x) = -\frac{1}{6}x + \frac{17}{16}$.

$$\therefore f'\left(\tfrac{1}{8}\left[3 - \sqrt{3081}\right]\right) \approx 2.16 \text{ and } f'\left(\tfrac{1}{8}\left[3 + \sqrt{3081}\right]\right) \approx -0.156$$

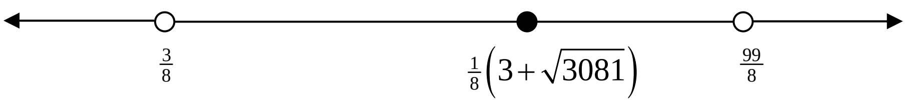
(f) $\therefore \left| f'\left(\tfrac{1}{8}\left[3 - \sqrt{3081}\right]\right)\right| > 1$, and so the fixed point at $x = \frac{1}{8}\left(3 - \sqrt{3081}\right)$ is repelling.

Also $\left| f'\left(\tfrac{1}{8}\left[3 + \sqrt{3081}\right]\right)\right| < 1$, and so the fixed point at $x = \frac{1}{8}\left(3 + \sqrt{3081}\right)$ is attracting.

(g) Yes. Each iteration sequence that we have considered either diverges or converges to the attracting fixed point at $x = \frac{1}{8}\left(3 + \sqrt{3081}\right)$

(h)
$$\begin{aligned} |f'(x)| < 1 &\Rightarrow -1 < f'(x) < 1 \\ &\Rightarrow -1 < -\frac{1}{6}x + \frac{17}{16} < 1 \\ &\Rightarrow -48 < -8x + 51 < 48 \\ &\Rightarrow -99 < -8x < -3 \\ &\Rightarrow \tfrac{3}{8} < x < \tfrac{99}{8} \end{aligned}$$

Hence the set of values of x for which $|f'(x)| < 1$ is the open interval $\left(\frac{3}{8}, \frac{99}{8}\right)$.



Since the attracting fixed point at $x = \frac{1}{8}\left(3 + \sqrt{3081}\right)$ is nearer to the endpoint $x = \frac{99}{8}$ than to the endpoint $x = \frac{3}{8}$, we choose *I* to have right-hand endpoint $x = \frac{99}{8}$. In order that the attracting fixed point at $x = \frac{1}{8}\left(3 + \sqrt{3081}\right)$ is the midpoint of *I*, to satisfy the gradient criterion, we take the left-hand endpoint to be