MST 121: Supplementary resource material for Chapter A1, Sequences

1. For each of the following sequences

   (a) 1, 4, 9, 16, 25…
   (b) 1, 3, 6, 10, 15, …
   (c) 1, 8, 27, 64, 125…

   - Write down the next 5 terms of the sequence
   - Write down an expression for the closed form of the sequence, taking the range variable as $n = 1,2,3,....$
   - If you have a graphical calculator, use the *seq* command to check that your expression for the closed form correctly generates the sequence.
   - Use your expression for the closed form to predict the 1000th term .
   - What special name is given to the terms of each sequence?

2. For each of the following arithmetic sequences

   (a) 5, 9, 13, 17, 21…
   (b) 11, 19, 27, 35, 43…
   (c) 4, 3, 2, 1, 0…
   (d) $-10, -4, 2, 8, 14, ...$
   (e) 2, 23, 44, 65, 86…

   - If you have a graphical calculator, devise a means of generating each of the above sequences by repeatedly pressing the **ENTER** key.
   - Write down the next 5 terms of each sequence
   - Define the arithmetic sequence as a recurrence system with starting value u_1
   - Write down an expression for the closed form of the sequence
   - If you have a graphical calculator, use the *seq* command to check that your expression for the closed form correctly generates the sequence.
   - Use your expression for the closed form to predict the 1000th term of each sequence.

3. For each of the following arithmetic sequences

   (a) 7, 9, 11, 13, 15…
   (b) 12, 19, 26, 33, 40…
   (c) 79, 72, 65, 58, 51…
   (d) 14, 10, 6, 2, -2, ….
   (e) 4, 15, 26, 37, 48…

   - If you have a graphical calculator, devise a means of generating each of the above sequences by repeatedly pressing the **ENTER** key.
   - Write down the next 5 terms of each sequence
   - Define the arithmetic sequence as a recurrence system with starting value u_0
   - Write down an expression for the closed form of the sequence
   - If you have a graphical calculator, use the *seq* command to check that your expression for the closed form correctly generates the sequence.
   - Use your expression for the closed form to predict the 1000th term of each sequence.

4. For each of the following geometric sequences

(a) 1, 2, 4, 8, 16…
(b) 3, 9, 27, 81, 243…
(c) 1, $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$, ….
(d) $216, 36, 6, 1, \frac{1}{6}, \frac{1}{36}$, ….
(e) x, xy, xy^2, xy^3, xy^4, ….

- ♦ If you have a graphical calculator, devise a means of generating each of the sequences (a) – (d) by repeatedly pressing the **ENTER** key.
- ♦ Write down the next 5 terms of each sequence
- ♦ Define the geometric sequence as a recurrence system with starting value u_1, and range variable $n = 1,2,3,...$
- ♦ Write down an expression for the closed form of each sequence.
- ♦ If you have a graphical calculator, use the *seq* command to check that your expression for the closed form correctly generates the sequence.
- ♦ Use the closed form to determine the 15th term of each sequence.
- ♦ Find the sum of the first fifteen terms of each sequence
- ♦ Describe the long-term behaviour of each of the sequences.

5. For each of the following geometric sequences

(a) 5, 10, 20, 40, 80, …
(b) 3, 12, 48, 192, 768…
(c) 6, 3, $\frac{3}{2}, \frac{3}{4}, \frac{3}{8}, \frac{3}{16}$, , ….
(d) 4, 20, 100, 500, 2500…
(e) $x, xy^2, xy^4, xy^6, xy^8$, ….

- ♦ If you have a graphical calculator, devise a means of generating each of the sequences (a)-(d) by repeatedly pressing the **ENTER** key.
- ♦ Write down the next 5 terms of the sequence
- ♦ Define the geometric sequence as a recurrence system with starting value u_0
- ♦ Write down an expression for the closed form of the sequence, with $n = 0,1,2,...$
- ♦ If you have a graphical calculator, use the *seq* command to check that your expression for the closed form correctly generates the sequence.
- ♦ Use your expression for the closed form to predict the 15th term of the sequence.
- ♦ Find the sum of the first fifteen terms of each sequence
- ♦ Describe the long-term behaviour of each of the sequences.

6. The general linear recurrence $x_0 = a,\ x_{n+1} = rx_n + d$ ($n = 0,1,2,...$ and $r \neq 1$) has a closed form $x_n = s \cdot r^n + t$, where $s = a + \dfrac{d}{r-1}$ and $t = -\dfrac{d}{r-1}$. Use this result to write down an expression for the closed form of each of the sequences given below, using $n = 0, 1, 2, ...$ as the range variable. In each case use the given recurrence system to generate the first 5 terms of the sequence, and check that your closed form produces the same sequence. Used the closed form to evaluate x_{20}.

(a) $x_{n+1} = 3x_n - 2,\ x_0 = 3,\ n = 0,1,2,....$
(b) $x_{n+1} = 2x_n + 5,\ x_0 = -2,\ n = 0,1,2,....$
(c) $x_{n+1} = 4x_n + 3,\ x_0 = 4,\ n = 0,1,2,....$

(d) $x_{n+1} = 7x_n - 18$, $x_0 = 7$, $n = 0,1,2,\ldots$

(e) $x_{n+1} = 9x_n + 344$, $x_0 = -41$, $n = 0,1,2,\ldots$

7. The general linear recurrence $x_1 = a$, $x_{n+1} = rx_n + d$ ($n = 1,2,3,\ldots$ and $r \neq 1$) has a closed form $x_n = s \cdot r^{n-1} + t$, where $s = a + \dfrac{d}{r-1}$ and $t = -\dfrac{d}{r-1}$. Use this result to write down an expression for the closed form of each of the sequences given below, using $n = 1, 2, 3, \ldots$ as the range variable. In each case use the given recurrence system to generate the first 5 terms of the sequence, and check that your closed form produces the same sequence. Used the closed form to evaluate x_{20}.

(a) $x_{n+1} = 5x_n - 1$, $x_1 = 7$, $n = 1,2,3,\ldots$

(b) $x_{n+1} = 2x_n + 11$, $x_1 = 5$, $n = 1,2,3,\ldots$

(c) $x_{n+1} = 3x_n - 2$, $x_1 = 9$, $n = 1,2,3,\ldots$

(d) $x_{n+1} = -4x_n + 20$, $x_1 = 79$, $n = 1,2,3,\ldots$

(e) $x_{n+1} = 9x_n - 14$, $x_1 = 23$, $n = 1,2,3,\ldots$

8. Taking the range variable as $n = 1,2,3,\ldots$, find a closed form for each of the following linear recurrences:

(a) 5, 17, 41… (b) 11, 17, 29… (c) 9, 74, 529… (d) 14, 24, 54…

Miscellaneous Problems:

9. The first three terms of a geometric progression are respectively the first, ninth and eleventh terms of an arithmetic progression. Given that the terms of the geometric progression are all different, find the value of the common ratio, *r*.

10. Given that $(x-1), (x+1)$ and $(x+9)$ are consecutive terms of a geometric sequence, determine the value of the common ratio of the sequence.

11. A sequence of numbers $\{u_n\}$ is defined, for $n \geq 1$, by the recurrence relation $u_{n+1} = ku_n - 4$, where *k* is a constant. Given that $u_1 = 2$

(a) Find expressions in terms of k for u_2 and u_3

(b) Given also that $u_3 = 26$, find the possible values of *k*.

(c) Given also that the sequence oscillates, calculate the value of u_4.

12. A sequence of numbers $t_1, t_2, t_3, t_4, \ldots$ is formed by taking a starting value of t_1 and using the rule $t_{k+1} = t_k^2 - 2$ for $k = 1,2,3,\ldots$

(a) If $t_1 = \sqrt{2}$, calculate t_2, t_3 and t_4. Show that $t_5 = 2$ and write down the value of t_{100}.

(b) If $t_1 = 2$ show that all the terms of the sequence are the same, and find the other value of t_1 for which all the terms are the same.

(c) Discuss the long term behaviour of the sequence in the cases when

(i) $t_1 = 3$ (ii) $t_1 = 1$ (iii) $t_1 = \dfrac{\sqrt{5}-1}{2}$

Answers

1. (a) ♦ 36, 49, 64, 81, 100 ♦ $u_n = n^2$, $n = 1,2,3,\ldots$ ♦ $seq(x^2, x, 1, 10)$
   ♦ $u_{1000} = 1000000$ ♦ square numbers

   (b) ♦ 21, 28, 36, 45, 55 ♦ $u_n = \frac{1}{2}n(n+1)$, $n = 1,2,3,\ldots$ ♦ $seq(\frac{1}{2}x(x+1), x, 1, 10)$
   ♦ $u_{1000} = 500500$ ♦ triangular numbers

   (c) ♦ 216, 343, 512, 729, 1000 ♦ $u_n = n^3$, $n = 1,2,3,\ldots$ ♦ $seq(x^3, x, 1, 10)$
   ♦ $u_{1000} = 1000000000 = 10^9$ ♦ cubes

2. (a) ♦ 5 ENTER + 4 ENTER ENTER ENTER… ♦ 25, 29, 33, 37, 41
   ♦ $u_{n+1} = u_n + 4, u_1 = 5, n = 1,2,3,\ldots$ ♦ $u_n = 4n + 1$, $n = 1,2,3,\ldots$
   ♦ $seq(4x + 1, x, 1, 10)$ ♦ $u_{1000} = 4001$

   (b) ♦ 11 ENTER + 8 ENTER ENTER ENTER… ♦ 51, 59, 67, 75, 83
   ♦ $u_{n+1} = u_n + 8, u_1 = 11, n = 1,2,3,\ldots$ ♦ $u_n = 8n + 3$, $n = 1,2,3,\ldots$
   ♦ $seq(8x + 3, x, 1, 10)$ ♦ $u_{1000} = 8003$

   (c) ♦ 4 ENTER –1 ENTER ENTER ENTER… ♦ -1, -2, -3, -4, -5
   ♦ $u_{n+1} = u_n - 1, u_1 = 4, n = 1,2,3,\ldots$ ♦ $u_n = 5 - n$, $n = 1,2,3,\ldots$
   ♦ $seq(5 - x, x, 1, 10)$ ♦ $u_{1000} = -995$

   (d) ♦ (-)10 ENTER +6 ENTER ENTER ENTER… ♦ 20, 26, 32, 38, 44
   ♦ $u_{n+1} = u_n + 6, u_1 = -10, n = 1,2,3,\ldots$ ♦ $u_n = 6n - 16$, $n = 1,2,3,\ldots$
   ♦ $seq(6x - 16, x, 1, 10)$ ♦ $u_{1000} = 5984$

   (e) ♦ 2 ENTER +21 ENTER ENTER ENTER… ♦ 107, 128, 149, 170, 191
   ♦ $u_{n+1} = u_n + 21, u_1 = 2, n = 1,2,3,\ldots$ ♦ $u_n = 21n - 19$, $n = 1,2,3,\ldots$
   ♦ $seq(21x - 19, x, 1, 10)$ ♦ $u_{1000} = 20981$

3. (a) ♦ 7 ENTER + 2 ENTER ENTER ENTER… ♦ 17, 19, 21, 23, 25
   ♦ $u_{n+1} = u_n + 2, u_0 = 7, n = 0,1,2,\ldots$ ♦ $u_n = 2n + 7$, $n = 0,1,2,\ldots$
   ♦ $seq(2x + 7, x, 0, 9)$ ♦ 1000^{th} term $= u_{999} = 2005$

   (b) ♦ 12 ENTER + 7 ENTER ENTER ENTER… ♦ 47, 54, 61, 68, 75, …
   ♦ $u_{n+1} = u_n + 7, u_0 = 12, n = 0,1,2,\ldots$ ♦ $u_n = 7n + 12$, $n = 0,1,2,\ldots$
   ♦ $seq(7x + 12, x, 0, 9)$ ♦ 1000^{th} term $= u_{999} = 7005$

   (c) ♦ 79 ENTER –7 ENTER ENTER ENTER… ♦ 44,37,30,23,16
   ♦ $u_{n+1} = u_n - 7, u_0 = 79, n = 0,1,2,\ldots$ ♦ $u_n = 79 - 7n$, $n = 0,1,2,\ldots$
   ♦ $seq(79 - 7x, x, 0, 9)$ ♦ 1000^{th} term $= u_{999} = -6914$

   (d) ♦ 14 ENTER –4 ENTER ENTER ENTER… ♦ $-6, -10, -14, -18, -22$
   ♦ $u_{n+1} = u_n - 4, u_0 = 14, n = 0,1,2,\ldots$ ♦ $u_n = 14 - 4n$, $n = 0,1,2,\ldots$
   ♦ $seq(14 - 4x, x, 0, 9)$ ♦ 1000^{th} term $= u_{999} = -3982$

   (e) ♦ 4 ENTER + 11 ENTER ENTER ENTER… ♦ 59, 70, 81, 92, 103
   ♦ $u_{n+1} = u_n + 11, u_0 = 4, n = 0,1,2,\ldots$ ♦ $u_n = 11n + 4$, $n = 0,1,2,\ldots$
   ♦ $seq(11x + 4, x, 0, 9)$ ♦ 1000^{th} term $= u_{999} = 10993$

4. (a) ♦ 1 ENTER × 2 ENTER ENTER ENTER… ♦ 32, 64, 128, 256, 512
   ♦ $u_{n+1} = 2u_n, u_1 = 1, n = 1,2,3,\ldots$ ♦ $u_n = 2^{n-1}$, $n = 1,2,3,\ldots$

♦ $seq(2^{x-1}, x, 1, 10)$ ♦ 15^{th} term $= 16384$ ♦ $s_{15} = 32767$ ♦ Diverges

(b) ♦ 3 **ENTER** $\times 3$ **ENTER** **ENTER** **ENTER**… ♦ 729, 2187, 6561, 19683, 59049

♦ $u_{n+1} = 3u_n, u_1 = 3, n = 1,2,3,...$ ♦ $u_n = 3^n$, $n = 1,2,3,....$

♦ $seq(3^x, x, 1, 10)$ ♦ 15^{th} term $= 14348907$ ♦ $s_{15} = 21523359$ ♦ Diverges

(c) ♦ 1 **ENTER** $\times \frac{1}{2}$ **ENTER** **ENTER** **ENTER**… ♦ $\frac{1}{32}, \frac{1}{64}, \frac{1}{128}, \frac{1}{256}, \frac{1}{512}$

♦ $u_{n+1} = \frac{1}{2}u_n, u_1 = 1, n = 1,2,3,...$ ♦ $u_n = 2^{1-n}$, $n = 1,2,3,....$

♦ $seq(2^{1-x}, x, 1, 10)$ ♦ 15^{th} term $= \frac{1}{16384}$ ♦ $s_{15} = \frac{32767}{16384}$ ♦ Converges to 0

(d) ♦ 216 **ENTER** $\times \frac{1}{6}$ **ENTER** **ENTER** **ENTER**… ♦ $\frac{1}{216}, \frac{1}{1296}, \frac{1}{7776}, \frac{1}{46656}, \frac{1}{279936}$

♦ $u_{n+1} = \frac{1}{6}u_n, u_1 = 216, n = 1,2,3,...$ ♦ $u_n = 6^{4-n}$, $n = 1,2,3,....$

♦ $seq(6^{4-x}, x, 1, 10)$ ♦ 15^{th} term $= \frac{1}{362797056} = 6^{-11}$

♦ $s_{15} = \frac{94036996915}{362797056} \approx 259.2$ ♦ Converges to 0

(e) ♦ [TI-89 users only] **x** **ENTER** $\times$ **y** **ENTER** **ENTER** **ENTER**…
$xy^5, xy^6, xy^7, xy^8, xy^9$ ♦ $u_{n+1} = yu_n, u_1 = x, n = 1,2,3...$ ♦ $u_n = xy^{n-1}$, $n = 1,2,...$

♦ [TI-89 users only] $seq(x \times y^{z-1}, z, 1, 10)$ ♦ 15^{th} term $= xy^{14}$ ♦ $s_{15} = \dfrac{x(1-y^{15})}{1-y}$

♦ Diverges if $y > 1$; Constant sequence for $y = 1$; Converges to 0 if $0 < y < 1$; Constant sequence for $y = 0$; Oscillates between positive and negative values, but converges to 0 if $-1 < y < 0$; Forms a two cycle between $\pm x$ if $y = -1$; Oscillates between positive and negative values and diverges for $y < -1$.

5. (a) ♦ 5 **ENTER** $\times 2$ **ENTER** **ENTER** **ENTER**… ♦ 160, 320, 640, 1280, 2560

♦ $u_{n+1} = 2u_n, u_0 = 5, n = 0,1,2,...$ ♦ $u_n = 5 \cdot 2^n$, $n = 0,1,2,....$

♦ $seq(5 \times 2^x, x, 0, 9)$ ♦ 15^{th} term $= u_{14} = 81920$ ♦ $s_{15} = 163835$ ♦ Diverges

(b) ♦ 3 **ENTER** $\times 4$ **ENTER** **ENTER** **ENTER**… ♦ 3072, 12288, 49152, 196608, 786432 ♦ $u_{n+1} = 4u_n$, $u_0 = 3$, $n = 0,1,2,...$ ♦ $u_n = 3 \cdot 4^n$, $n = 0,1,2,...,$

♦ $seq(3 \times 4^x, x, 0, 9)$ ♦ 15^{th} term $= u_{14} = 805306368$ ♦ $s_{15} = 1073741823$

♦ Diverges

(c) ♦ 6 **ENTER** $\times \frac{1}{2}$ **ENTER** **ENTER** **ENTER**… ♦ $\frac{3}{32}, \frac{3}{64}, \frac{3}{128}, \frac{3}{256}, \frac{3}{512}$

♦ $u_{n+1} = \frac{1}{2}u_n$, $u_0 = 6$, $n = 0,1,2,...$ ♦ $u_n = 6 \cdot (\frac{1}{2})^n$, $n = 0,1,2,...,$

♦ $seq(6 \times (\frac{1}{2})^x, x, 0, 9)$ ♦ 15^{th} term $= u_{14} = \frac{3}{8192}$ ♦ $s_{15} = \frac{98301}{8192}$

♦ Converges to 0

(d) ♦ 4 **ENTER** $\times 5$ **ENTER** **ENTER** **ENTER**… ♦ 12500, 62500, 312500, 1562500, 7812500 ♦ $u_{n+1} = 5u_n$, $u_0 = 4$, $n = 0,1,2,...$

♦ $u_n = 4 \cdot 5^n$, $n = 0,1,2,...,$ ♦ $seq(4 \times 5^x, x, 0, 9)$

♦ 15^{th} term $= u_{14} = 24414062500$ ♦ $s_{15} = 30517578124$ ♦ Diverges

(e) ♦ [TI-89 users only] **x** **ENTER** $\times$ **y** $\wedge 2$ **ENTER** **ENTER** **ENTER**…

♦ $xy^{10}, xy^{12}, xy^{14}, xy^{16}, xy^{18}$ ♦ $u_{n+1} = y^2u_n$, $u_0 = x$, $n = 0,1,2...$

♦ $u_n = xy^{2n}$, $n = 0,1,2,...$ ♦ [TI-89 users only] $seq(x \times y^{2z}, z, 0, 9)$

♦ 15^{th} term $= u_{14} = xy^{28}$ ♦ $s_{15} = \dfrac{x(1-y^{30})}{1-y^2}$ ♦ Diverges if $|y| > 1$ and converges to 0 if $|y| < 1$; constant sequence for $y = 0,1$.

6. (a) $x_n = 2\cdot 3^n + 1, n = 0,1,2,...$; 3 ENTER 3Ans -2 ENTER ENTER ENTER…;
3, 7, 19, 55, 163 ; $seq(2\times 3^x + 1, x, 0, 4)$; $x_{20} = 6973568803$

(b) $x_n = 3\cdot 2^n - 5, n = 0,1,2,...$; -2 ENTER 2Ans +5 ENTER ENTER ENTER…;
-2, 1, 7, 19, 43 ; $seq(3\times 2^x - 5, x, 0, 4)$; $x_{20} = 3145723$

(c) $x_n = 5\cdot 4^n - 1,\ n = 0,1,2,...$; 4 ENTER 4Ans +3 ENTER ENTER ENTER…;
4, 19, 79, 319, 1279 ; $seq(5\times 4^x - 1, x, 0, 4)$; $x_{20} = 5497558138879$

(d) $x_n = 4\cdot 7^n + 3,\ n = 0,1,2,...$; 7 ENTER 7Ans -18 ENTER ENTER ENTER...;
7, 31, 199, 1375, 9607 ; $seq(4\times 7^x + 3, x, 0, 4)$; $x_{20} = 319169065190448007$

(e) $x_n = 2\cdot 9^n - 43,\ n = 0,1,2,...$; (-) 41 ENTER 9Ans +344 ENTER ENTER …;
$-41, -25, 119, 1415, 13079$; $seq(2\times 9^x - 43, x, 0, 4)$; $x_{20} = 24315330918113857559$

7. (a) $x_n = \frac{1}{4}(27\cdot 5^{n-1} + 1), n = 1,2,3,.$; 7 ENTER 5Ans$-1$ ENTER ENTER ENTER..
7, 34, 169, 844, 4219 ; $seq(0.25(27\times 5^{x-1} + 1), x, 1, 5)$;
$x_{20} = 128746032714844 \approx 1.287\times 10^{14}$

(b) $x_n = 16\cdot 2^{n-1} - 11, n = 1,2,3,.$; 5 ENTER 2Ans +11 ENTER ENTER ENTER..;
5, 21, 53, 117, 245, …; $seq(16\times 2^{x-1} - 11, x, 1, 5)$; $x_{20} = 8388597$

(c) $x_n = 8\cdot 3^{n-1} + 1, n = 1,2,3,...$; 9 ENTER 3Ans -2 ENTER ENTER ENTER..;
9, 25, 73, 217, 649 ; $seq(8\times 3^{x-1} + 1, x, 1, 5)$; $x_{20} = 9298091737$

(d) $x_n = 75\cdot(-4)^{n-1} + 4, n = 1,2,3,...$; 79 ENTER (-4) Ans +20 ENTER ENTER..;
79, -296, 1204, -4796, 19204 ; $seq\ (75\times(-4)^{x-1} + 4, x, 1, 5)$
$x_{20} = -20615843020796 \approx -2.06\times 10^{13}$

(e) $x_n = \frac{1}{4}(85\cdot 9^{n-1} + 7), n = 1,2,3,.$; 23 ENTER 9Ans -14 ENTER ENTER..;
23, 193, 1723, 15493, 139423 ; $seq(\frac{1}{4}(85\times 9^{x-1} + 7), x, 1, 5)$;
$x_{20} = 28705599000551081893 \approx 2.87\times 10^{19}$

8. (a) $x_n = 12\times 2^{n-1} - 7, n = 1,2,3,...$ (b) $x_n = 6\times 2^{n-1} + 5, n = 1,2,3,...$
(c) $x_n = \frac{1}{6}(65\times 7^{n-1} - 11), n = 1,2,3,...$ (d) $x_n = 5\times 3^{n-1} + 9, n = 1,2,3,...$

9. Eliminate d from $a + 8d = ar$ and $a + 10d = ar^2$ to give $4r^2 - 5r + 1 = 0$. Reject $r = 1$ to give $r = \frac{1}{4}$.

10. Solve $\dfrac{x+1}{x-1} = \dfrac{x+9}{x+1}$ to give $x = \frac{5}{3}$. Then $r = \dfrac{\frac{5}{3}+1}{\frac{5}{3}-1} = 4$.

11. (a) $u_2 = 2k - 4,\ u_3 = 2k^2 - 4k - 4$ (b) $-3, 5$ (c) Taking $k = -3$ gives $u_4 = -82$

12. (a) $t_2 = 0,\ t_3 = -2,\ t_4 = 2,\ t_5 = 2^2 - 2 = 2,\ t_{100} = 2$

(b) With $t_1 = 2,\ t_2 = 2^2 - 2 = 2$. Hence all terms will be 2. Put $t_{k+1} = t_k = t$ to give $t = t^2 - 2$ with roots -1 and 2. Hence the other value of t is -1.

(c) (i) The sequence diverges

(ii) $t_i = -1$ for $i \geq 2$

(iii) The sequence gives a 2-cycle with values $\frac{1}{2}\left(-1 \pm \sqrt{5}\right)$