MST 121: Supplementary Resource Material for Chapter A0, Starting Points

- **Powers, Indices and Standard Form**

1. Which of 8^3 and 8×10^3 is the larger, and by how much?

2. Without using a calculator, express each of the following powers as a single fraction in its lowest terms:

   (a) $36^{\frac{1}{2}}$ (b) $27^{\frac{1}{3}}$ (c) $81^{-\frac{3}{4}}$ (d) $125^{-\frac{2}{3}}$ (e) $\left(\frac{27}{64}\right)^{-\frac{2}{3}}$ (f) $\left(\frac{121}{144}\right)^{-\frac{3}{2}}$ (g) $243^{\frac{2}{5}}$

   In each case please check your answers using a calculator.

3. Simplify each of the following expressions:

   (a) $x^{2y}\times x^{3y}$ (b) $3^{24}\div3^{22}$ (c) $(3p)^0$ (d) $(x^3)^{2y}$ (e) $\frac{3^{1-2x}}{3^{2-2x}}$ (f) $\sqrt{x}\times\sqrt[3]{x}\times x^{\frac{1}{6}}$

   (g) $2a^2b\times3ab^3\times4ab$ (h) $8x\times3xy^2\times\frac{1}{8}x^{-2}\times y$ (i) $3^{1-2p}\times3^{2p+1}\times9^{3p+5}$

   (j) $\frac{4a^3b^2c^5}{2a^2b^2c^6}$ (k) $\frac{3x^3y^4z^2}{2x^5}\times\frac{8x^4y^2z}{9xyz^2}$ (l) $\frac{p^2q^3r^{1-s}}{3pq}\div\frac{q^{-2}r^{2-s}}{9pr}$ (m) $\frac{\sqrt{x}\times\sqrt[5]{x}}{\sqrt[3]{x}\div x^{-1}}$

4. A *Fermat Number* is defined to be a number of the form $F_n=2^{2^n}+1$, where *n* is an integer. If F_n is prime, then it is said to be a *Fermat prime.* Fermat believed that F_n was prime for all positive integers, *n*. However, in 1732 the Swiss mathematician Leonhard Euler (1707-1783) discovered a factorisation of F_5. Confirm F_5 is composite by verifying that it is equal to 641×6700417.

5. Newton's Law of gravitation states that the force, *F*, exerted on one body by another is given by $F=\frac{Gm_1m_2}{r^2}$ where m_1 and m_2 are the respective masses, *r* is their distance apart and *G* is a constant called the *universal constant of gravitation.*

   Use Newton's Law to calculate the force exerted on the moon by the Earth. [You may assume that the mass of the Earth is 5.98×10^{24} kg, the mass of the moon is 7.34×10^{22} kg, the mean radius of the moon's orbit around the Earth is 3.8×10^8 m and the value of the gravitational constant *G* is 6.67×10^{-11} $Nm^2\,kg^{-2}$.] Give your answer in Newtons, to one significant figure.

6. In 1991, astronomers at the Palomar Observatory using a 200 inch reflector discovered a quasar 12 billion light years away. Given that a billion is 10^9, and that light travels at 299800 kms^{-1}, express the distance to this quasar in metres. Take 1 year as 365.25 days and give your answer to two significant figures.

7. Srinivasa Ramanujan (1887-1920) was a Hindu mathematician with great talent. He discovered that $\sqrt{\sqrt{\frac{2143}{22}}}$ was an extremely good approximation to π. Find, to two significant figures, the percentage error in this approximation.

- **Formula**

8. A quadrilateral inscribed in a circle is called a *cyclic quadrilateral*.

If such a quadrilateral has sides of length a, b, c and d, then its area is given by

$$A = \sqrt{(s-a)(s-b)(s-c)(s-d)},$$

where $s = \frac{1}{2}(a+b+c+d)$ is the semi-perimeter of the cyclic quadrilateral.
Use this result to show that the area of a cyclic quadrilateral with sides of length 6 cm, 8 cm, 10 cm and $\sqrt{57}-3$ cm is 46.5 cm^2, to 3 significant figures.

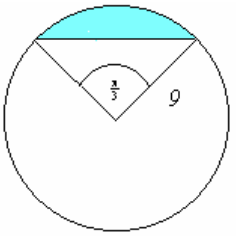
9. The area of an arbitrary quadrilateral with sides of length a, b, c and d is given by $A = \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd\cos^2\alpha}$ where $s = \frac{1}{2}(a+b+c+d)$ and $\alpha = \frac{1}{2}$ the sum of either of the two pairs of opposite angles.

(i) What is the sum of the opposite angles of a cyclic quadrilateral?
(ii) Prove that the general formula given above reduces to

$$A = \sqrt{(s-a)(s-b)(s-c)(s-d)}$$

when the quadrilateral is cyclic.

10. The area of a segment of a circle of radius r, subtending θ radians at the centre, is given by $A = \frac{1}{2}r^2(\theta - \sin\theta)$.



Use this formula to calculate the ***exact*** area of a segment of a circle with radius 9 and subtending $\frac{\pi}{3}$ radians at the centre.

11. The mass of a decaying radioactive isotope is given by $m = 800e^{-kt}$, where m is the mass in grams and t is the time in years.

(a) What is the initial mass of the isotope?
(b) Given that the half life of the isotope is 100 years, what is the value of k (to 3 significant figures)?
(c) What will be the mass of the isotope after 500 years?
(d) How many complete years will it take for the mass of the isotope to be reduced to less than 1 gram?

- **Sets**

12. Which of the sets $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Q}$, or $\mathbb{R}$ (if any) is the smallest set that each of the following numbers must necessarily belong to?

(a) 4 (b) π (c) 0 (d) π^3 (e) $\frac{19}{31}$ (f) -1 (g) $\sqrt[5]{17\pi^3}$ (h) $1-7\sqrt{2}$ (i) $\sqrt{-1}$

- **Algebra**

13. Expand each of the following expressions:

(a) $(x+y)(2x-3y)$ (b) $(x+y)^2$ (c) $(x+y)^3$ (d) $(x-2)^2(y+1)^2$

(e) $\dfrac{x^2-7xy+xy^5}{x}$ (f) $\dfrac{3y-4y^3-5x^2y^2}{y}$ (g) $\dfrac{a^3b^2-ab^4+ab}{ab}$

14. Factorise each of the following expressions completely:

(a) $9x+12y$ (b) $\pi r^2+2\pi rh$ (c) $4-8x^2$ (d) $2+4x-12x^2$ (e) $1-4b^2$

(f) $x^2y^2-9y^4$ (g) $a^2-9a^2b^2$ (h) $n-n^3$ (i) $4q^2-25$ (j) $\pi^3-\pi^5$

(k) $x^2+7x+12$ (l) $x^2-12x+20$ (m) a^2+a-6 (n) $p^2-5p-14$

(o) $4x^2-8x-5$ (p) $12x^2-7x+1$ (q) $12x^2+7x-12$ (r) $18x^2+15x+2$

(s) $ax+bx-ay-by$ (t) $3pq+12p+2q+8$ (u) $4tx-14t-10x+35$

15. Express each of the following as single fractions in their lowest terms:

(a) $\frac{2x-5}{x+7}+\frac{x-3}{x-1}$ (b) $\frac{x+1}{x-1}-\frac{x-1}{x+1}$ (c) $\frac{x-1}{3(2x+5)}-\frac{7(x+2)}{x-3}$ (d) $1-\frac{x-1}{x+1}$

(e) $\frac{x^2+5x+6}{x^2+5x+4}\times\frac{x^2+7x+12}{x^2-3x-10}$ (f) $\frac{1}{x^2-1}-\frac{1}{x+1}$ (g) $\frac{x^2+9x+14}{x^2+2x-3}\div\frac{x+7}{x+3}$

16. Solve each of the following linear equations:

(a) $4x-7=2x+11$ (b) $8x-5=3x+35$ (c) $4(x+2)+5=3x$ (d) $5(x\text{-}1)+8=3x$

(e) $\frac{7}{x-9}=\frac{3}{x+2}$ (f) $\frac{x-6}{5}=\frac{3x-7}{11}$ (g) $\frac{x+7}{3}+\frac{x-2}{4}-\frac{x+3}{5}-\frac{x+11}{7}=-1$

17. Solve each of the following simultaneous equations by eliminating one of the variables. Give each of your answers as exact fractions.

(a) $\begin{aligned}3x+4y&=-7\\2x+9y&=5\end{aligned}$ (b) $\begin{aligned}5x-11y&=0\\3x+13y&=-5\end{aligned}$ (c) $\begin{aligned}8x+17y&=-1\\9x-7y&=-23\end{aligned}$

18. Solve each of the following simultaneous equations by substituting for one of the variables. Give each of your answers as exact fractions.

(a) $\begin{aligned}7x+y&=-9\\5x-8y&=2\end{aligned}$ (b) $\begin{aligned}x-17y&=1\\2x+19y&=0\end{aligned}$ (c) $\begin{aligned}2x-11y&=11\\5x-y&=14\end{aligned}$

19. Solve each of the following quadratic equations by firstly factorising them into a product of two linear factors:

(a) $x^2+5x-14=0$ (b) $x^2-6x-16=0$ (c) $x^2+9x-10=0$

20. Solve each of the following quadratic equations by substituting the coefficients into the formula $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$. You should leave each of your answers in exact surdform.

(a) $x^2-x-1=0$ (b) $7x^2-11x-13=0$ (c) $5x^2+9x-8=0$

21. Solve each of the following quadratic equations by firstly completing the square. You should again leave each of your answers in exact surdform.

(a) $x^2-6x-1=0$ (b) $x^2+8x-5=0$ (c) $3x^2+12x-7=0$

22. Find the exact values of x that satisfy each of the following equations:

(a) $\frac{x-3}{x-5}=\frac{2x-7}{5x+3}$ (b) $1+\frac{1}{x-3}=\frac{x-11}{7}$ (c) $x^3-5x^2-19x=2x$

Answers

1. 8×10^3, by 7488

2 (a) 6 (b) 3 (c) $\frac{1}{27}$ (d) $\frac{1}{25}$ (e) $\frac{16}{9}$ (f) $\frac{1728}{1331}$ (g) 9

3. (a) x^{5y} (b) 3^2 (c) 1 (d) x^{6y} (e) $\frac{1}{3}$ (f) x (g) $24a^4b^5$ (h) $3y^3$ (i) 3^{6p+12}
(j) $\frac{2a}{c}$ (k) $\frac{4}{3}xy^5z$ (l) $3p^2q^4$ (m) $x^{-\frac{19}{30}}$

4. $2^{32}+1 = 4294967297 = 641\times6700417$

5. $2\times10^{20}\ N$

6. $1.1\times10^{26}\ m$

7. 3.2×10^{-8} %

9. (i) 180^0
(ii) If the quadrilateral is cyclic, then its opposite angles each add up to 180^0. Hence $\cos^2\frac{1}{2}\alpha = \cos^2 90 = 0$ (as $\cos 90 = 0$), and so the formulae reduces to that given for a cyclic quadrilateral.

10. $\frac{27}{4}(2\pi - 3\sqrt{3})$

11. (a) 800 grams (b) 0.00693 (c) 25 grams (d) 965 years

12. (a) $\mathbb{N}$ (b) $\mathbb{R}$ (c) $\mathbb{Z}$ (d) $\mathbb{R}$ (e) $\mathbb{Q}$ (f) $\mathbb{Z}$ (g) $\mathbb{R}$ (h) $\mathbb{R}$ (i) None of the given sets

13. (a) $2x^2 - xy - 3y^2$ (b) $x^2 + 2xy + y^2$ (c) $x^3 + 3x^2y + 3xy^2 + y^3$
(d) $x^2y^2 - 4xy^2 + 4y^2 + 2x^2y - 8xy + 8y + x^2 - 4x + 4$ (e) $x - 7y + y^5$
(f) $3 - 4y^2 - 5x^2y$ (g) $a^2b - b^3 + 1$

14. (a) $3(3x+4y)$ (b) $\pi r(r+2h)$ (c) $4(1-2x^2)$ (d) $2(1+2x-6x^2)$
(e) $(1+2b)(1-2b)$ (f) $y^2(x+3y)(x-3y)$ (g) $a^2(1+3b)(1-3b)$
(h) $n(1+n)(1-n)$ (i) $(2q+5)(2q-5)$ (j) $\pi^3(1+\pi)(1-\pi)$ (k) $(x+3)(x+4)$
(l) $(x-2)(x-10)$ (m) $(a-2)(a+3)$ (n) $(p+2)(p-7)$ (o) $(2x+1)(2x-5)$
(p) $(4x-1)(3x-1)$ (q) $(4x-3)(3x+4)$ (r) $(6x+1)(3x+2)$ (s) $(a+b)(x-y)$
(t) $(3p+2)(q+4)$ (u) $(2x-7)(2t-5)$

15. (a) $\dfrac{3x^2-3x-16}{(x+7)(x-1)}$ (b) $\dfrac{4x}{(x+1)(x-1)}$ (c) $-\dfrac{41x^2+193x+207}{3(2x+5)(x-3)}$ (d) $\dfrac{2}{x+1}$
(e) $\dfrac{(x+3)^2}{(x+1)(x-5)}$ (f) $\dfrac{2-x}{(x+1)(x-1)}$ (g) $\dfrac{x+2}{x-1}$

16. (a) $x=9$ (b) $x=8$ (c) $x=-13$ (d) $x=-\frac{3}{2}$ (e) $x=-\frac{41}{4}$ (f) $x=-\frac{31}{4}$
(g) $x=-\frac{278}{101}$

17. (a) $x=-\frac{83}{19},\ y=\frac{29}{19}$ (b) $x=-\frac{55}{98},\ y=-\frac{25}{98}$ (c) $x=-\frac{398}{209},\ y=\frac{175}{209}$

18. (a) $x=-\frac{70}{61}, y=-\frac{59}{61}$ (b) $x=\frac{19}{53}, y=-\frac{2}{53}$ (c) $x=\frac{143}{53}, y=-\frac{27}{53}$

19. (a) $x=-7, 2$ (b) $x=-2, 8$ (c) $x=-10, 1$

20. (a) $x=\frac{1}{2}(1\pm\sqrt{5})$ (b) $x=\frac{1}{14}(11\pm\sqrt{485})$ (c) $x=\frac{1}{10}(-9\pm\sqrt{241})$

21. (a) $x=3\pm\sqrt{10}$ (b) $x=-4\pm\sqrt{21}$ (c) $x=\frac{1}{3}(-6\pm\sqrt{57})$

22. (a) $x=\frac{1}{6}(-5\pm\sqrt{553})$ (b) $x=\frac{1}{2}(21\pm\sqrt{253})$ (c) $x=0, \frac{1}{2}(5\pm\sqrt{109})$