

MS221: Tutorial Support Material for Chapter A3, Functions from geometry

1. Write down a parametrisation function for each of the following curves:

(a) The circle $(x + 4)^2 + (y - 2)^2 = 9$ (b) The parabola $y^2 = 20x$

(c) The left branch of the hyperbola $\frac{1}{5}x^2 - \frac{1}{12}y^2 = 1$

In each case substitute your proposed parametrisation into the given Cartesian equation to confirm that your solution is correct.

2. This question concerns the quadratic curve L with equation $x^2 - y^2 - 8x + 10y = 10$

(a) Find the translation function that maps the quadratic curve L onto a conic K in standard position. Write down the equation of K .

(b) Write down the translation function that maps K onto L

(c) Find a parametrisation function (or functions) for the quadratic curve, L

3. Express each of the following translations in the form $t_{p,q}$

(a) The inverse of $t_{-2,8}$ (b) the composite $t_{6,-1} \circ t_{2,-2}$

4. For each of the following angles, use the rotation formula

$$r_\theta : (x, y) \mapsto (x \cos \theta - y \sin \theta, x \sin \theta + y \cos \theta)$$

to determine the function that specifies the rotation through that angle about the origin. In each case

- ◆ Calculate the image of $(1,0)$ under the given function
- ◆ Determine the inverse function
- ◆ Check that the inverse function correctly maps the image of $(1,0)$ calculated above back onto $(1,0)$

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{6}$ (d) $-\frac{3\pi}{2}$

5. Use the result $r_\theta \circ r_\phi = r_{\theta+\phi}$, to determine each of the following composite functions. In each case, please ensure that your answer lies in the interval $(-\pi, \pi]$.

(a) $r_{\frac{\pi}{3}} \circ r_{\frac{\pi}{6}}$ (b) $r_{\frac{\pi}{4}} \circ r_{-\frac{\pi}{6}}$ (c) $r_{\frac{2\pi}{3}} \circ r_{\frac{3\pi}{4}}$ (d) $r_{-\frac{\pi}{2}} \circ r_{-\frac{5\pi}{6}}$

6. Consider the reflection q and translation t with definitions given by

$$q : \mathfrak{R}^2 \rightarrow \mathfrak{R}^2 \quad \text{and} \quad t : \mathfrak{R}^2 \rightarrow \mathfrak{R}^2$$
$$(x, y) \mapsto (-x, y) \quad \text{and} \quad (x, y) \mapsto (x, y - 5)$$

- (a) Describe the geometric effects of the functions q and t .
- (b) What special name is given to the composite of these two functions
- (c) Determine the composite functions $q \circ t$ and $t \circ q$, and comment on the result.
- (d) Use the definitions of q and t to determine the image of $(2,3)$, and verify that $q \circ t(2,3) = t \circ q(2,3)$.

7. Let q be the transformation that reflects the plane in the y -axis, and let r be the transformation that rotates the plane clockwise about the origin through $\frac{\pi}{6}$ radians. Algebraically, these have the forms

$$q : \mathfrak{R}^2 \rightarrow \mathfrak{R}^2 \quad r : \mathfrak{R}^2 \rightarrow \mathfrak{R}^2$$

$$(x, y) \mapsto (-x, y) \quad (x, y) \mapsto \left(\frac{1}{2}(\sqrt{3}x - y), \frac{1}{2}(x + \sqrt{3}y) \right)$$

- (d) Determine the composite isometries $r \circ q$ and $q \circ r$.
- (e) is it true that $r \circ q = q \circ r$?
- (f) Use your answer to (b) to make a general statement about combinations of reflections and rotations.
8. Let l be the line through the origin that makes an angle θ radians with the direction of the positive x -axis. For each of the following values of θ , use the function

$$q_\theta : \mathfrak{R}^2 \rightarrow \mathfrak{R}^2$$

$$(x, y) \mapsto (x \cos(2\theta) + y \sin(2\theta), x \sin(2\theta) - y \cos(2\theta))$$

to determine the image of an arbitrary point (x, y) under the reflection q_θ

- (a) $\theta = \frac{\pi}{8}$ (b) $\theta = \frac{\pi}{12}$ (c) $\theta = \frac{\pi}{4}$ (d) $\theta = \frac{2\pi}{3}$
9. (a) Find $\sec \theta$ and $\cot \theta$, given that $\sin \theta = \frac{5}{13}$ and $\theta \in (\frac{\pi}{2}, \pi)$
- (b) Find $\operatorname{cosec} \theta$ and $\cos \theta$, given that $\tan \theta = \frac{3}{4}$ and $\theta \in (\pi, \frac{3\pi}{2})$
- (c) Find $\sin \theta$, $\sec \theta$ and $\operatorname{cosec} \theta$, given that $\cot \theta = \frac{8}{15}$ and $\theta \in (0, \frac{\pi}{2})$
10. Use the addition formulas
- $$\sin(\theta \pm \phi) = \sin \phi \cos \theta \pm \cos \phi \sin \theta$$
- $$\cos(\theta \pm \phi) = \cos \phi \cos \theta \mp \sin \phi \sin \theta$$
- to find the exact values of
- (a) $\sin \frac{5\pi}{12}$ (b) $\operatorname{cosec} \frac{5\pi}{12}$ (c) $\sec \frac{\pi}{12}$ (d) $\cos \frac{7\pi}{12}$
11. Use $\tan(\theta \pm \phi) = \frac{\tan \theta \pm \tan \phi}{1 \mp \tan \theta \tan \phi}$ to find the exact values of (a) $\tan \frac{5\pi}{12}$ (b) $\cot \frac{7\pi}{12}$
12. Use the half angle formulas $\sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta)$ and $\cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta)$ to find the exact values of (a) $\sin \frac{3\pi}{8}$ and (b) $\cos \frac{3\pi}{8}$.
- and hence deduce the exact values of (c) $\tan \frac{3\pi}{8}$ (d) $\operatorname{cosec} \frac{3\pi}{8}$ (e) $\sec \frac{3\pi}{8}$ (f) $\cot \frac{3\pi}{8}$
13. Use the half angle formulas to find $\sin \theta$ and $\cos \theta$, where θ is the angle in the interval $(0, \frac{\pi}{2})$ for which $\cos(2\theta) = -\frac{5}{9}$.
14. Use the techniques introduced in section 4 of the unit to sketch the curves with equations $14x^2 + 5xy + 2y^2 - 28 = 0$ and $5x^2 + 8xy - 10y^2 - 12 = 0$

Answers:

1. (a) This curve is a circle, centre $(-4,2)$ and radius 3. Hence

$$p : [0, 2\pi] \rightarrow \mathfrak{R}^2$$

$$t \mapsto (3 \cos t - 4, 3 \sin t + 2)$$

- (b) This curve is a parabola in standard position, $y^2 = 4ax$, with $a = 5$. Hence

$$p : \mathfrak{R} \rightarrow \mathfrak{R}^2$$

$$t \mapsto (5t^2, 10t)$$

- (g) This curve is a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ in standard position with $a = \sqrt{5}$ and $b = \sqrt{12} = 2\sqrt{3}$. Using $(x, y) = (a \sec t, b \tan t)$ gives the parametrisation of the left branch of the hyperbola as

$$p : \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \rightarrow \mathfrak{R}^2$$

$$t \mapsto \left(\sqrt{5} \sec t, 2\sqrt{3} \tan t\right)$$

2. (a) $x^2 - y^2 - 8x + 10y = 10 \Rightarrow (x-4)^2 - (y-5)^2 = 1$. So the transformation t is

$$t : \mathfrak{R}^2 \rightarrow \mathfrak{R}^2$$

$$(x, y) \mapsto (x-4, y-5)$$

The image of L under t is the hyperbola K with equation $x^2 - y^2 = 1$

- (b) $t : \mathfrak{R}^2 \rightarrow \mathfrak{R}^2$

$$(x, y) \mapsto (x+4, y+5)$$

- (c) The rules are the same for each branch of the hyperbola, but the domains of the parameter t are not. The parametrisations of the left and right branches of the hyperbola are respectively

$$p : \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \rightarrow \mathfrak{R}^2 \quad \text{and} \quad p : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \rightarrow \mathfrak{R}^2$$

$$t \mapsto (4 + \sec t, 5 + \tan t) \quad t \mapsto (4 + \sec t, 5 + \tan t)$$

3. (a) $t_{2,-8}$ (b) $t_{8,-3}$

4. (a) ♦ $r_{\frac{\pi}{2}} : \mathfrak{R}^2 \rightarrow \mathfrak{R}^2$ ♦ $(1,0) \mapsto (0,1)$ ♦ $r_{-\frac{\pi}{2}} : \mathfrak{R}^2 \rightarrow \mathfrak{R}^2$
 $(x, y) \mapsto (-y, x)$ $(x, y) \mapsto (y, -x)$

- (b) ♦ $r_{\frac{\pi}{3}} : \mathfrak{R}^2 \rightarrow \mathfrak{R}^2$ ♦ $(1,0) \mapsto \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
 $(x, y) \mapsto \left(\frac{1}{2}(x - \sqrt{3}y), \frac{1}{2}(\sqrt{3}x + y)\right)$

- ♦ $r_{-\frac{\pi}{3}} : \mathfrak{R}^2 \rightarrow \mathfrak{R}^2$
 $(x, y) \mapsto \left(\frac{1}{2}(x + \sqrt{3}y), \frac{1}{2}(y - \sqrt{3}x)\right)$

$$(c) \quad \blacklozenge \quad r_{\frac{\pi}{6}} : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \quad \blacklozenge \quad (1,0) \mapsto \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$$

$$(x, y) \mapsto \left(\frac{1}{2}(\sqrt{3}x - y), \frac{1}{2}(x + \sqrt{3}y)\right)$$

$$\blacklozenge \quad r_{-\frac{\pi}{6}} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

$$(x, y) \mapsto \left(\frac{1}{2}(\sqrt{3}x + y), \frac{1}{2}(\sqrt{3}y - x)\right)$$

(d) The results here are the same as those obtained in (a), for a rotation through $\frac{\pi}{2}$ radians anticlockwise is equivalent to a rotation through $\frac{3\pi}{2}$ radians clockwise

$$\blacklozenge \quad r_{\frac{\pi}{2}} : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \quad \blacklozenge \quad (1,0) \mapsto (0,1) \quad \blacklozenge \quad r_{-\frac{\pi}{2}} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

$$(x, y) \mapsto (-y, x) \quad (x, y) \mapsto (y, -x)$$

5. (a) $r_{\frac{\pi}{2}}$ (b) $r_{\frac{\pi}{12}}$ (c) $r_{\frac{17\pi}{12}} = r_{-\frac{7\pi}{12}}$ (d) $r_{-\frac{4\pi}{3}} = r_{\frac{2\pi}{3}}$

6. (a) q describes reflection in the y -axis, and t describes a vertical translation through 5 units downwards.

(b) Glide reflection

$$(c) \quad t \circ q : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \quad \text{and} \quad q \circ t : \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

$$(x, y) \mapsto (-x, y - 5) \quad (x, y) \mapsto (-x, y - 5)$$

It does not matter in which order we form the composite, as $t \circ q = q \circ t$

(d) $(2,3) \mapsto (-2, -2)$

7. (a) $r \circ q : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $q \circ r : \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$$(x, y) \mapsto \left(-\frac{1}{2}(\sqrt{3}x + y), \frac{1}{2}(\sqrt{3}y - x)\right) \quad (x, y) \mapsto \left(-\frac{1}{2}(\sqrt{3}x - y), \frac{1}{2}(x + \sqrt{3}y)\right)$$

(b) No. $r \circ q \neq q \circ r$

(c) Composition of rotations and reflections is not commutative. In other words, it matters in which order the transformations are composed.

8. (a) $q_{\frac{\pi}{8}} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ (b) $q_{\frac{\pi}{12}} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$$(x, y) \mapsto \left(\frac{1}{\sqrt{2}}(x + y), \frac{1}{\sqrt{2}}(x - y)\right) \quad (x, y) \mapsto \left(\frac{1}{2}(\sqrt{3}x + y), \frac{1}{2}(x - \sqrt{3}y)\right)$$

(c) $q_{\frac{\pi}{4}} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ (d) $q_{\frac{2\pi}{3}} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$$(x, y) \mapsto (y, x) \quad (x, y) \mapsto \left(-\frac{1}{2}(x + \sqrt{3}y), \frac{1}{2}(y - \sqrt{3}x)\right)$$

9. (a) $\sec \theta = -\frac{13}{12}$ and $\cot \theta = -\frac{12}{5}$ (b) $\operatorname{cosec} \theta = -\frac{5}{3}$ and $\cos \theta = -\frac{4}{5}$

(c) $\sin \theta = \frac{15}{17}$, $\sec \theta = \frac{17}{8}$ and $\operatorname{cosec} \theta = \frac{17}{15}$

10. (a) $\frac{1}{4}(\sqrt{2} + \sqrt{6})$ (b) $\sqrt{6} - \sqrt{2}$ (c) $\sqrt{6} - \sqrt{2}$ (d) $\frac{1}{4}(\sqrt{2} - \sqrt{6})$

11. (a) $2 + \sqrt{3}$ (b) $\sqrt{3} - 2$

12. (a) $\sin\left(\frac{3\pi}{8}\right) = \sqrt{\frac{1 - \cos\frac{3\pi}{4}}{2}} = \sqrt{\frac{1 - (-\frac{1}{\sqrt{2}})}{2}} = \frac{1}{2}\sqrt{2 + \sqrt{2}}$ (b) $\frac{1}{2}\sqrt{2 - \sqrt{2}}$

$$(c) \frac{\sqrt{2+\sqrt{2}}}{\sqrt{2-\sqrt{2}}} = \sqrt{3+2\sqrt{2}} = 1+\sqrt{2} \quad (d) \frac{2}{\sqrt{2+\sqrt{2}}} \quad (e) \frac{2}{\sqrt{2-\sqrt{2}}} \quad (f) \frac{1}{1+\sqrt{2}} = 1+\sqrt{2}$$

13. Since $\theta \in (0, \frac{\pi}{2})$, we know that $\sin \theta$ and $\cos \theta$ are both positive. Hence

$$\sin \theta = \sqrt{\frac{1-\cos 2\theta}{2}} = \sqrt{\frac{1+\frac{5}{9}}{2}} = \sqrt{\frac{7}{9}} = \frac{1}{3} \cdot \sqrt{7}$$

$$\cos \theta = \sqrt{\frac{1+\cos 2\theta}{2}} = \sqrt{\frac{1-\frac{5}{9}}{2}} = \sqrt{\frac{2}{9}} = \frac{1}{3} \cdot \sqrt{2}$$

14. • $14x^2 + 5xy + 2y^2 - 28 = 0$ is of the general form $Ax^2 + Bxy + Cy^2 + F = 0$ where $A = 14, B = 5, C = 2$ and $F = -28$.

• As $A \neq C$, we have $\tan 2\theta = \frac{B}{A-C} = \frac{5}{12}$. Hence the angle of inclination, θ , is given by $\theta = \frac{1}{2} \arctan \frac{5}{12} \approx 11^\circ$

• We must now apply equations 4.6 on page 40 of Unit A3. Firstly, however, we note that $\tan 2\theta = \frac{5}{12} \Rightarrow \sin 2\theta = \frac{5}{13}$ and $\cos 2\theta = \frac{12}{13}$

$$\therefore \cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta) = \frac{1}{2}\left(1 + \frac{12}{13}\right) = \frac{25}{26}$$

$$\sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta) = \frac{1}{2}\left(1 - \frac{12}{13}\right) = \frac{1}{26}$$

$$\sin \theta \cos \theta = \frac{1}{2} \sin 2\theta = \frac{1}{2} \times \frac{5}{13} = \frac{5}{26}$$

$$\therefore A' = A \cos^2 \theta + B \sin \theta \cos \theta + C \sin^2 \theta = 14 \times \frac{25}{26} + 5 \times \frac{5}{26} + 2 \times \frac{1}{26} = \frac{29}{2}$$

$B' = 0$ (by construction, as our intention is to eliminate the term in xy)

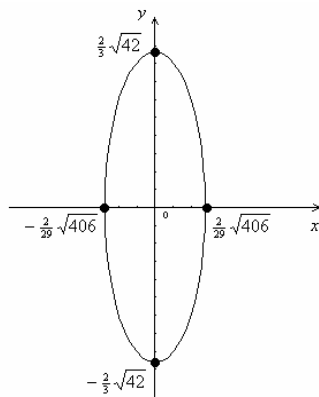
$$C' = A \sin^2 \theta - B \sin \theta \cos \theta + C \cos^2 \theta = 14 \times \frac{1}{26} - 5 \times \frac{5}{26} + 2 \times \frac{25}{26} = \frac{3}{2}$$

$D' = E' = 0$, and $F' = F = -28$

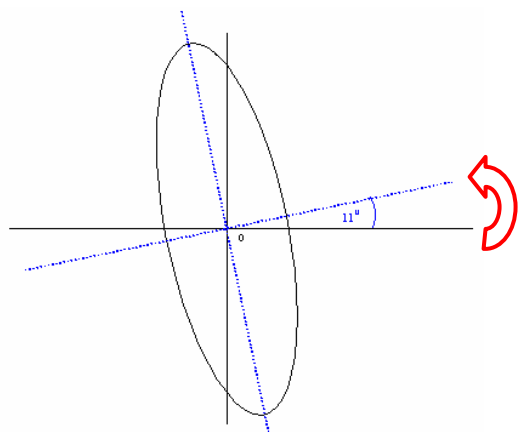
• Hence the equation of the conic in standard position is

$$A'x^2 + C'y^2 + F' = 0 \Rightarrow \frac{29}{2}x^2 + \frac{3}{2}y^2 - 28 = 0$$

This simplifies to $29x^2 + 3y^2 = 56$, which is an ellipse that we can easily sketch:



To sketch the given conic, we must rotate the ellipse on the left through 11° anticlockwise.



- ◆ $5x^2 + 8xy - 10y^2 - 12 = 0$ is of the general form $Ax^2 + Bxy + Cy^2 + F = 0$ where $A = 5, B = 8, C = -10$ and $F = -12$.
- ◆ As $A \neq C$, we have $\tan 2\theta = \frac{B}{A-C} = \frac{8}{15}$. Hence the angle of inclination, θ , is given by $\theta = \frac{1}{2} \arctan \frac{8}{15} \approx 14^\circ$

- ◆ We must now apply equations 4.6 on page 40 of Unit A3. Firstly, however, we note that $\tan 2\theta = \frac{8}{15} \Rightarrow \sin 2\theta = \frac{8}{17}$ and $\cos 2\theta = \frac{15}{17}$

$$\therefore \cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta) = \frac{1}{2}\left(1 + \frac{15}{17}\right) = \frac{16}{17}$$

$$\sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta) = \frac{1}{2}\left(1 - \frac{15}{17}\right) = \frac{1}{17}$$

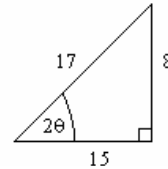
$$\sin \theta \cos \theta = \frac{1}{2} \sin 2\theta = \frac{1}{2} \times \frac{8}{17} = \frac{4}{17}$$

$$\therefore A' = A \cos^2 \theta + B \sin \theta \cos \theta + C \sin^2 \theta = 5 \times \frac{16}{17} + 8 \times \frac{4}{17} - 10 \times \frac{1}{17} = 6$$

$$B' = 0 \text{ (by construction, as our intention is to eliminate the term in } xy\text{)}$$

$$C' = A \sin^2 \theta - B \sin \theta \cos \theta + C \cos^2 \theta = 5 \times \frac{1}{17} - 8 \times \frac{4}{17} - 10 \times \frac{16}{17} = -11$$

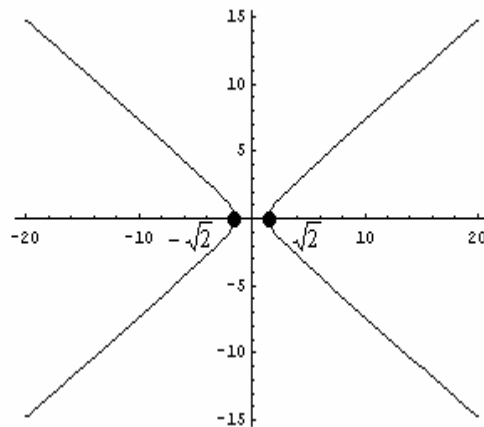
$$D' = E' = 0, \text{ and } F' = F = -12$$



- ◆ Hence the equation of the conic in standard position is

$$A'x^2 + C'y^2 + F' = 0 \Rightarrow 6x^2 - 11y^2 - 12 = 0$$

This is a hyperbola in standard position, which we can easily sketch:



To sketch the given conic, we must rotate the one in standard position through 14° anticlockwise.

