MS221: Tutorial Support Material for Chapter A2, Conics

1. Re-arrange the following equations in the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, and hence sketch the corresponding ellipses:

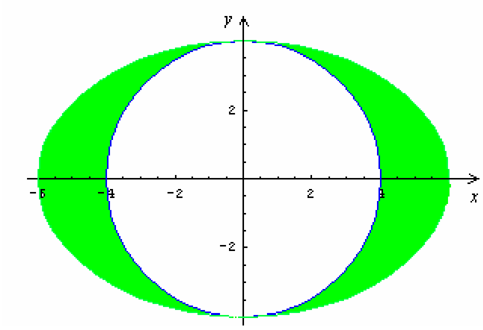
   (a) $x^2+2y^2=8$ (b) $2x^2+y^2=18$ (c) $3x^2+4y^2=4$

   Which of these three ellipses is not in standard position?

2. On the same sketch draw the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$ and its bounding circle $x^2+y^2=25$. Describe the transformations by which

   (a) The bounding circle can be transformed into the ellipse
   (b) The ellipse can be transformed into the bounding circle

3. The diagram on the right shows the ellipse $\frac{x^2}{36}+\frac{y^2}{16}=1$, along with its inscribed circle $x^2+y^2=16$. Given that the area of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is πab, show that $\frac{1}{3}$ of the area of the ellipse is shaded.



4. On the same sketch draw the ellipse $\frac{x^2}{49}+\frac{y^2}{25}=1$ and its inscribed circle $x^2+y^2=25$. Describe the transformations by which

   (a) The bounding circle can be transformed into the ellipse
   (b) The ellipse can be transformed into the bounding circle

5. Sketch the hyperbola $4x^2-25y^2=81$, and include on your sketch the equations of the asymptotes and any points of intersection with the co-ordinate axes.

6. For each of the following ellipses, find the foci, directrices and eccentricity, and mark the foci and directrices on a sketch of the ellipse:

   (a) $4x^2+9y^2=100$ (b) $x^2+5y^2=25$

7. Find the focus and directrix of each of the following parabolas, and indicate these on a sketch of the parabola.

   (a) $y^2=16x$ (b) $12x-y^2=0$

   Write down the eccentricity of each of the two parabolas in (a) and (b).

8. Find the foci, directrices and asymptotes of each of the following hyperbolas, and include these on a sketch of each hyperbola.

(a) $x^2 - 16 - 2y^2 = 0$ (b) $9y^2 - 36x^2 + 1 = 0$

State, with reasons, whether or not either of these hyperbolas are rectangular hyperbolas.

9. Prove that the eccentricity of any rectangular hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $\sqrt{2}$.

10. Classify each of the following conic sections as a circle, ellipse, parabola or hyperbola, and then sketch each curve. Include on your sketch, as appropriate, the vertices, axes of symmetry, foci, directrices and asymptotes, clearly labelling each with their equations or co-ordinates.

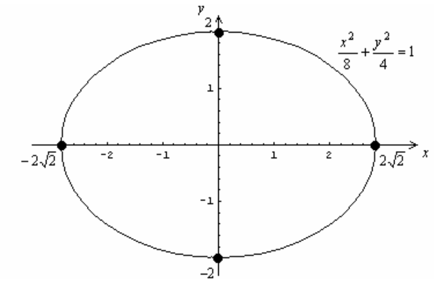
(a) $y^2 - 8x - 2y - 39 = 0$ (b) $x^2 - 4y^2 - 16x + 32y - 18 = 0$

(c) $x^2 + 2y^2 - 16x - 4y + 30 = 0$ (d) $x^2 + y^2 + 4x - 6y - 23 = 0$

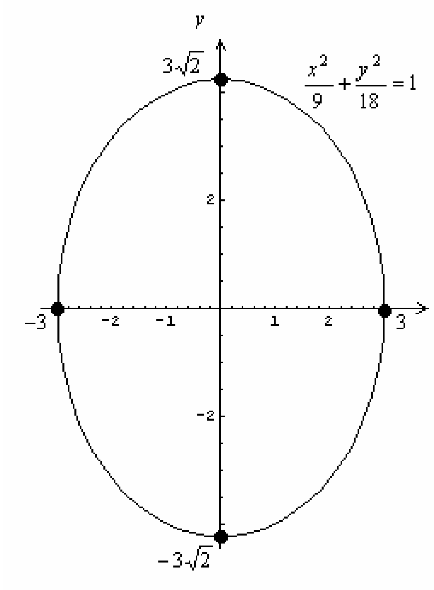
11. Write down the parametric equations for each of the quadratic curves in question 10.

Answers:

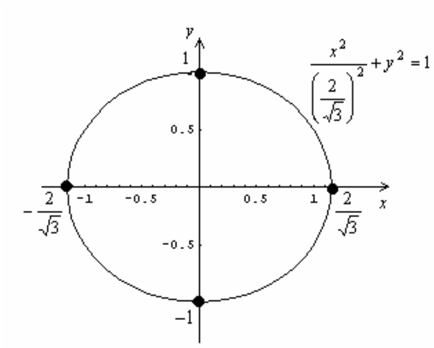
1. (a)



(b)

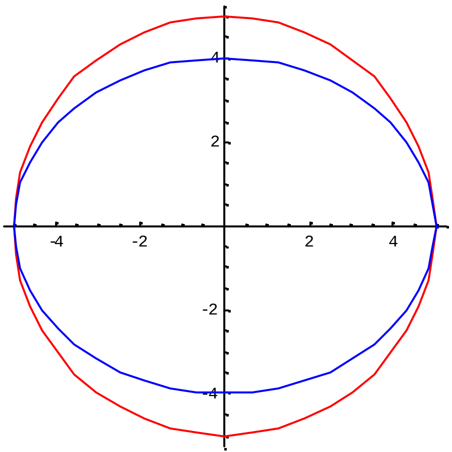


(c)



Ellipse (b) is not in standard position because $a < b$.

2.



The equation $\frac{x^2}{25}+\frac{y^2}{16}=1$ can be written as $x^2+\left(\frac{5}{4}y\right)^2=25$, whereby we can see that the point $\left(x,\frac{5}{4}y\right)$ lies on the bounding circle. Hence

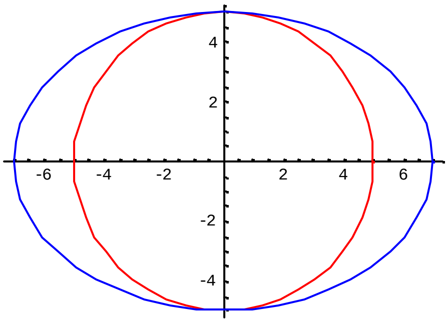
(a) The bounding circle can be transformed into the ellipse by a scaling in the y direction with scale factor $\left(\frac{5}{4}\right)^{-1}=\frac{4}{5}$

(b) The ellipse can be transformed into the bounding circle by a scaling in the y direction with scale factor $\frac{5}{4}$.

3. The area of the ellipse is $\pi\times 6\times 4=24\pi$, and the area of the circle is $\pi\times 4^2=16\pi$

$\therefore$ Shaded area $=24\pi-16\pi=8\pi$, and so the fraction of the ellipse shaded is $\frac{8\pi}{24\pi}=\frac{1}{3}$

4.

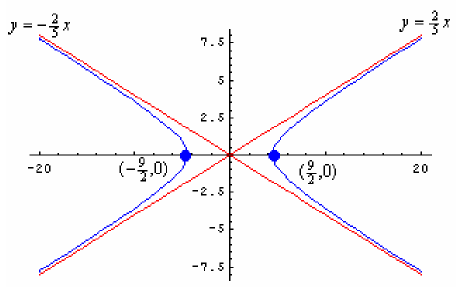


The equation $\frac{x^2}{49}+\frac{y^2}{25}=1$ can be written as $\left(\frac{5}{7}x\right)^2+y^2=25$, whereby we can see that the point $\left(\frac{5}{7}x,y\right)$ lies on the inscribed circle. Hence

(a) The inscribed circle can be transformed into the ellipse by a scaling in the x direction with scale factor $\left(\frac{5}{7}\right)^{-1}=\frac{7}{5}$

(b) The ellipse can be transformed into the inscribed circle by a scaling in the x direction with scale factor $\frac{5}{7}$.

5. We can re-arrange $4x^2 - 25y^2 = 81$ to give $\dfrac{x^2}{\left(\frac{9}{2}\right)^2} - \dfrac{y^2}{\left(\frac{9}{5}\right)^2} = 1$. This is now in the standard form $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, with $a = \frac{9}{2}$ and $b = \frac{9}{5}$. The equations of the asymptotes are $y = \pm\frac{2}{5}x$, and the curve cuts the x-axis at $x = \left(\pm\frac{9}{2}, 0\right)$



6. (a) The equation $4x^2 + 9y^2 = 100$ can be written in standard form as $\dfrac{x^2}{5^2} + \dfrac{y^2}{\left(\frac{10}{3}\right)^2} = 1$.

Hence $a = 5$ and $b = \frac{10}{3}$. The eccentricity is given by $e = \sqrt{1 - \left(\frac{b}{a}\right)^2} = \frac{1}{3}\sqrt{5}$

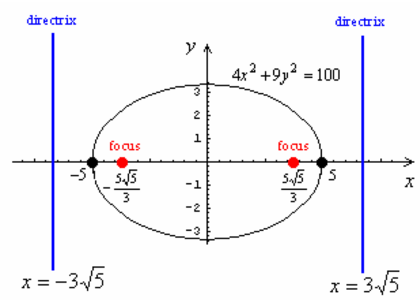
The foci are at $(\pm ae, 0) = \left(\pm\frac{5\sqrt{5}}{3}, 0\right)$ and the directrices at $x = \pm\frac{a}{e} = \pm 3\sqrt{5}$

(b) The equation $x^2 + 5y^2 = 25$ can be written in standard form as $\dfrac{x^2}{5^2} + \dfrac{y^2}{(\sqrt{5})^2} = 1$.

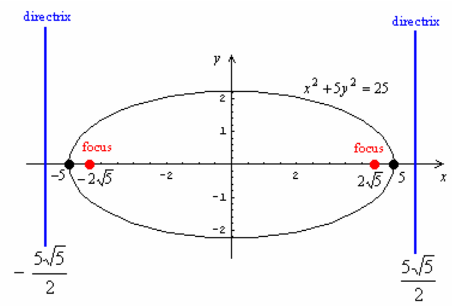
Hence $a = 5$ and $b = \sqrt{5}$. The eccentricity is given by $e = \sqrt{1 - \left(\frac{b}{a}\right)^2} = \frac{2\sqrt{5}}{5}$

The foci are at $(\pm ae, 0) = \left(\pm 2\sqrt{5}, 0\right)$ and the directrices at $x = \pm\frac{a}{e} = \pm\frac{5\sqrt{5}}{2}$

(a)



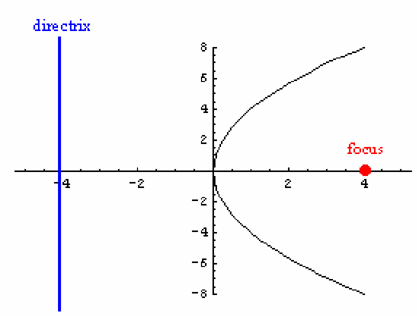
(b)



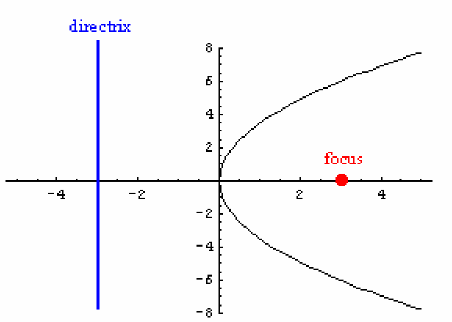
7. (a) $y^2 = 16x \Rightarrow y^2 = 4 \cdot 4x$. Hence the focus is at $(4,0)$ and the directrix is the line $x = -4$.

(b) $12x - y^2 = 0 \Rightarrow y^2 = 4 \cdot 3x$. Hence the focus is at $(3,0)$ and the directrix is the line $x = -3$.

(a)



(b)

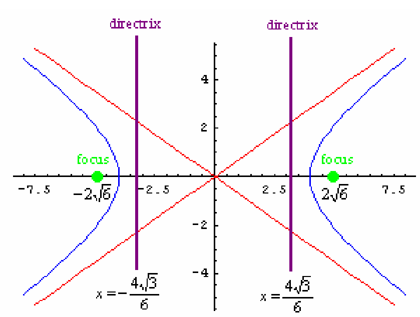


The eccentricity of each of the two parabolas is 1.

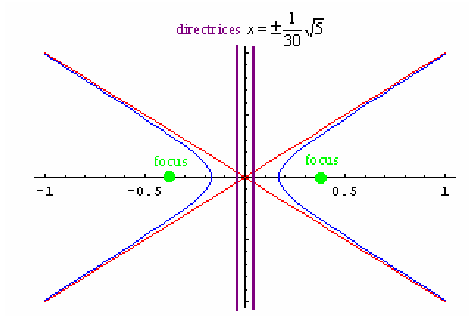
8. (a) We can re-arrange $x^2 - 16 - 2y^2 = 0$ to give $\dfrac{x^2}{16} - \dfrac{y^2}{8} = 1$, which is now in the standard form $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ with $a^2 = 16$ and $b^2 = 8$. Hence the eccentricity is given by $e = \sqrt{1 + \frac{b^2}{a^2}} = \frac{1}{2}\sqrt{6}$. The foci are at $(\pm ae, 0) = (\pm 2\sqrt{6}, 0)$, the directrices are $x = \pm\dfrac{a}{e} = \pm\dfrac{4}{3}\sqrt{6}$ and the equations of the asymptotes are $y = \pm\dfrac{b}{a}x = \pm\dfrac{1}{2}\sqrt{2}x$. Finally, the curve cuts the x-axis at $x = (\pm 4, 0)$.

(b) We can re-arrange $9y^2 - 36x^2 + 1 = 0$ to give $\dfrac{x^2}{(\frac{1}{6})^2} - \dfrac{y^2}{(\frac{1}{3})^2} = 1$, which is now in the standard form $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ with $a^2 = \dfrac{1}{36}$ and $b^2 = \dfrac{1}{9}$. Hence the eccentricity is given by $e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{5}$. The foci are at $(\pm ae, 0) = (\pm\frac{1}{6}\sqrt{5}, 0)$, the directrices are $x = \pm\dfrac{a}{e} = \pm\dfrac{1}{30}\sqrt{5}$ and the equations of the asymptotes are $y = \pm\dfrac{b}{a}x = 2x$. Finally, the curve cuts the x-axis at $x = (\pm\frac{1}{6}, 0)$.

(a)



(b)



Neither of the hyperbolas in (a) and (b) are rectangular hyperbolas because neither pair of asymptotes are perpendicular.

9. The asymptotes of the rectangular hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ are $y = \pm\dfrac{b}{a}x$. Now for a hyperbola to be classified as rectangular we require these asymptotes to intersect at right angles, and the condition for this is that the product of the gradients is -1.

$$\therefore \frac{b}{a} \times -\frac{b}{a} = -1 \Rightarrow b^2 = a^2$$

It follows that $\dfrac{b^2}{a^2} = 1$, and so the eccentricity is given by $e = \sqrt{1 + \dfrac{b^2}{a^2}} = \sqrt{1+1} = \sqrt{2}$.

10. (a) $y^2 - 8x - 2y - 39 = 0$ can be re-arranged as $(y-1)^2 = 4 \cdot 2(x+5)$. Hence this conic section is the parabola $y^2 = 8x$, translated through $\begin{pmatrix} -5 \\ 1 \end{pmatrix}$. It follows that the vertex is at $(-5, 1)$, the focus at $(-3, 1)$ and the directrix at $x = -7$. As for all parabolas, its eccentricity is 1.

(b) $x^2 - 4y^2 - 16x + 32y - 18 = 0$ can be re-arranged as $(x-8)^2 - 4(y-4)^2 - 18 = 0$.

Hence this conic section is the hyperbola $x^2 - 4y^2 = 18$, translated through $\begin{pmatrix} 8 \\ 4 \end{pmatrix}$.

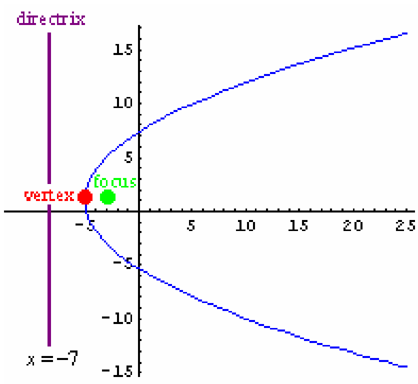
It follows that the eccentricity is $\frac{1}{2}\sqrt{5}$, the foci are at $\left(8 \pm \frac{3}{2}\sqrt{10}, 4\right)$ and the directrices are $x = 8 \pm \frac{6}{5}\sqrt{10}$. The asymptotes become $y = \frac{1}{2}x$ and $y = -\frac{1}{2}x + 8$.

(c) $x^2 + 2y^2 - 16x - 4y + 30 = 0$ can be re-arranged as $(x-8)^2 + 2(y-1)^2 = 36$.

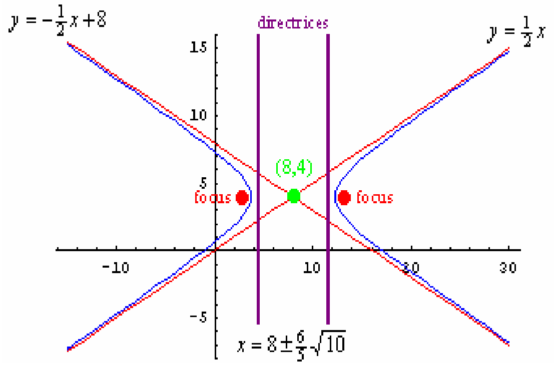
Hence this conic section is the ellipse $x^2 + 2y^2 = 36$, translated through $\begin{pmatrix} 8 \\ 1 \end{pmatrix}$. It follows that the centre is at $(8,1)$, the eccentricity is $\frac{1}{2}\sqrt{2}$, the foci are at $\left(8 \pm 3\sqrt{2}, 1\right)$ and the directrices are $x = 8 \pm 6\sqrt{2}$.

(d) $x^2 + y^2 + 4x - 6y - 23 = 0$ can be re-arranged as $(x+2)^2 + (y-3)^2 = 36$. Hence this quadratic curve is the circle $x^2 + y^2 = 36$ translated through $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$. It follows that its centre is at $(-2,3)$, and its radius is 6.

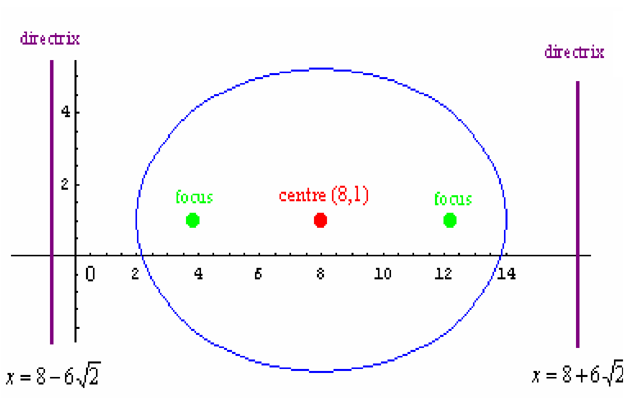
(a)



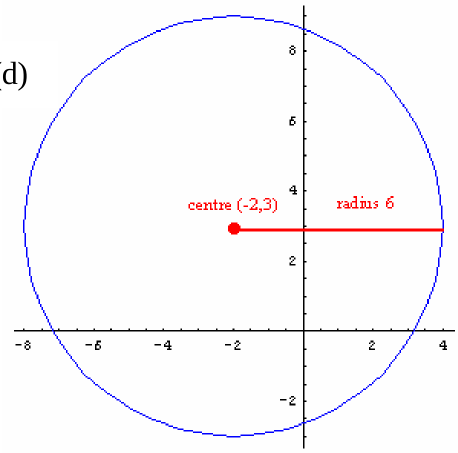
(b)



(c)



(d)



11. (a) $(x,y) = (2t^2 - 5, 4t + 1),\ t \in \Re$

(b) $(x,y) = (8 + 3\sqrt{2}\sec t, 4 + \frac{3\sqrt{2}}{2}\tan t),\ t \in (-\frac{\pi}{2}, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \frac{3\pi}{2})$

(c) $(x,y) = (8 + 6\cos t, 1 + 3\sqrt{2}\sin t),\ t \in (0, 2\pi)$

(d) $(x,y) = (6\cos t - 2, 6\sin t + 3),\ t \in (0, 2\pi)$