MS221: Supplementary Resource Material for Chapter A1, Exploring Sequences

1. Rationalise the denominator of each of the following fractions:

   (a) $\frac{4}{\sqrt{6}}$ (b) $\frac{2}{3-\sqrt{2}}$ (c) $\frac{5}{\sqrt{3}-1}$ (d) $\frac{2+\sqrt{8}}{2-\sqrt{8}}$ (e) $\frac{2-\sqrt{3}}{3-\sqrt{2}}$ (f) $\frac{\sqrt{6}+\sqrt{8}}{\sqrt{2}+\sqrt{10}}$

2. Suppose that the roots of the quadratic equation $ax^2+bx+c=0$ are α and β. Devise a method of showing that the sum and product of the roots are respectively $\alpha+\beta=-\frac{b}{a}$ and $\alpha\beta=\frac{c}{a}$, without actually solving the equation.

3. For each of the following quadratic equations, find the sum and the product of the roots without actually solving the equations:

   (a) $x^2+7x-5=0$ (b) $2x^2-17x+4=0$ (c) $3x^2-9x-11=0$

4. For each of the following quadratic equations, find the sum of the squares of the roots and the sum of the reciprocals of the roots without solving the equations:

   (a) $8x^2+2x-5=0$ (b) $6x^2-5x-11=0$ (c) $7x^2-4x+6=0$

   - Attempt to find the roots of the quadratic equation in (c). What do you notice, and how might you explain this?

5. Suppose that the roots of the quadratic equation $3x^2-7x+1=0$ are α and β. Without actually solving this equation, find those quadratic equations with roots

   (a) α^2 and β^2 (b) α^3 and β^3 (c) α^4 and β^4 (d) $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$

6. Suppose that the roots of the cubic equation $ax^3+bx^2+cx+d=0$ are α, β and γ. Use the method developed in question (2) to show the following relationships between the roots and the coefficients:

   $$\alpha+\beta+\gamma=-\frac{b}{a} \quad ; \quad \alpha\beta+\alpha\gamma+\beta\gamma=\frac{c}{a} \quad ; \quad \alpha\beta\gamma=-\frac{d}{a}$$

   Hence show that the sum and product of the roots of the cubic equation $5x^3-7x^2+11x-19=0$ are respectively $\frac{7}{5}$ and $\frac{19}{5}$.

7. By forming and solving the respective auxiliary equations, find a closed form for each of the following recurrence systems:

   (a) $u_0=3, u_1=7 \quad u_{n+2}=7u_{n+1}-12u_n \quad n=0,1,2,\ldots$

   (b) $u_0=8, u_1=5 \quad u_{n+2}=3u_{n+1}-2u_n \quad n=0,1,2,\ldots$

   (c) $u_0=10, u_1=20 \quad u_{n+2}=6u_{n+1}+4u_n \quad n=0,1,2,\ldots$

8. By forming and solving the respective auxiliary equations, find a closed form for each of the following recurrence systems:

   (a) $u_0=1, u_1=1 \quad u_{n+2}=10u_{n+1}-25u_n \quad n=0,1,2,\ldots$

(b) $u_0 = 1, u_1 = -1 \quad u_{n+2} = 12u_{n+1} - 36u_n \quad n = 0,1,2,...$

(c) $u_0 = 4, u_1 = 8 \quad u_{n+2} = 4u_n \quad n = 0,1,2,...$

9. The closed form for the Fibonacci numbers is $F_n = \frac{1}{\sqrt{5}}\left(f^n - y^n\right)$, $n = 1,2,3,...$, where $f = \frac{1}{2}(1+\sqrt{5})$ and $y = \frac{1}{2}(1-\sqrt{5})$ are the roots of the golden ratio equation $x^2 - x - 1 = 0$. Use this closed form to determine each of the following Fibonacci numbers, and in each case show that Binet's approximation (the nearest integer to $\frac{f^n}{\sqrt{5}}$) gives the same result.

(a) F_{10} (b) F_{20} (c) F_{30} (d) F_{40} (e) F_{50}

10. Exercise 3.2 in the text introduces the Lucas numbers, named after the French mathematician Francois Edouard Anatole Lucas (1841-1891). These are essentially a Fibonacci sequence generated by different starting values, and have a closed form $L_n = f^n + y^n \quad (n = 1,2,3,...)$.

The solution to exercise 3.2 gives the range variable as $n = 0,1,2,...$ but other texts often use $n = 1,2,3,...$. What difference, if any, does this make to the sequence?

11. The course material has introduced several identities on the Fibonacci numbers, but there are many more. Here are another two

♦ $F_2 + F_4 + F_6 + ... + F_{2n} = F_{2n+1} - 1$ ♦ $F_{2n} = F_{n+1}^2 - F_{n-1}^2$

(a) Write out the first 10 terms of the Fibonacci sequence and use these to convince yourselves that these identities are true.
(b) Use telescoping cancellation to prove that the first identity holds.
(c) Use the fact that f satisfies the golden ratio equation to show that $\frac{f^4 - 1}{f^2} = 1 + \frac{2}{f}$ and deduce that $\frac{f^4 - 1}{f^2} = \sqrt{5}$. Similarly, show that $\frac{y^4 - 1}{y^2} = -\sqrt{5}$.
(d) Use the results from (c) together with the closed form $F_n = \frac{1}{\sqrt{5}}\left(f^n - y^n\right)$ and the identity $(a-b)^2 \equiv a^2 - 2ab + b^2$ to prove that the second identity is true.

12. As the Fibonacci and Lucas sequences are generated by the same process, it is perfectly reasonable to suppose that there are many identities connecting them. Here are two such identities:

♦ $L_n = F_{n-1} + F_{n+1}$ ♦ $F_{2n} = F_n \cdot L_n$

Prove these identities by reference to the corresponding closed forms.

13. Use the closed forms for the Fibonacci and Lucas sequences to prove the Cassini type identities (a) $L_{2n} = L_n^2 - 2\cdot(-1)^n$ (b) $F_{3n} = F_n \cdot (L_{2n} + (-1)^n)$

14. Calculate f^n for $n = 1,2,...,9$, expressing your answers in the form $\frac{1}{2}(a_n\sqrt{5} + b_n)$.

What do you observe about the sequences a_n and b_n.

Miscellaneous questions:

15. Consider the recurrence system $u_0 = 1, u_1 = 4$ $\quad u_{n+2} = 2u_{n+1} - 4u_n$ $\quad n = 0,1,2,...$

- Use the recurrence relation to generate the sequence as far as u_{10}.
- Describe what happens when you formulate and attempt to solve the auxiliary equation.
- With your calculator set to radian mode, find the first 11 terms of the sequence $u_n = 2^n\left(\cos\frac{n\pi}{3} + \sqrt{3}\sin\frac{n\pi}{3}\right), n = 0,1,...$
- What do you observe?
- Formulate a conjecture for the closed form of the above recurrence system, and explain why this conjecture may not be correct.

16. Consider the recurrence relation $u_{n+1} = \frac{u_n^{\ 2}}{u_{n-1}}$, $n = 1,2,3,....$

Evaluate the terms u_2 to u_5 for each of the following pairs of starting values, and in each case state what kind of sequence is being generated.

(i) $u_0 = u_1 = 5$ (ii) $u_0 = 6, u_1 = 3$ (iii) $u_0 = 4, u_1 = 16$

Comment on the type of sequence you would expect the recurrence equation to generate for other pairs of starting values.

17. Consider the recurrence equation $u_{n+1} = 7u_n + 4u_{n-1}$, $n = 1,2,3,....$ with starting values $u_0 = 2$ and $u_1 = 5$. Let $A_n = \frac{u_n}{u_{n-1}}$.

(i) Evaluate the terms u_2 to u_5.

(ii) Evaluate the terms A_2 to A_5.

(iii) Prove that $A_{n+1} = 7 + \frac{4}{A_n}$

(iv) Show that the solution to the equation $A = 7 + \frac{4}{A}$ is $A = \frac{1}{2}\cdot\left(7 \pm \sqrt{65}\right)$

(v) How you would expect A_n to behave for large values of *n*.

18. Suppose that you wished to find a second order recurrence relation whose solution generated an arithmetic sequence. Show that the only recurrence relation that will achieve this is essentially given by $u_{n+1} = 2u_n - u_{n-1}$, $n = 1,2,3,.....$

Answers:

1. (a) $\frac{2}{3}\sqrt{6}$ (b) $\frac{2}{7}(3+\sqrt{2})$ (c) $\frac{5}{2}(1+\sqrt{3})$ (d) $-(3+2\sqrt{2})$

   (e) $\frac{1}{7}(6+2\sqrt{2}-3\sqrt{3}-\sqrt{6})$ (f) $\frac{1}{4}(\sqrt{15}+2\sqrt{5}-\sqrt{3}-2)$

2. As the roots of $ax^2+bx+c=0$ are α and β, then $(x-\alpha)(x-\beta)=0$ which expands to give $x^2-(\alpha+\beta)x+\alpha\beta=0$. If we now divide $ax^2+bx+c=0$ through by a to obtain $x^2+\frac{b}{a}x+\frac{c}{a}=0$, the result follows by comparing the coefficients of these two equivalent equations.

3. (a) sum $=-7$, product $=-5$ (b) sum $=\frac{17}{2}$, product $=2$

   (c) sum $=3$, product $=-\frac{11}{3}$

4. (a) sum of squares $=\frac{21}{16}$, sum of reciprocals $=\frac{2}{5}$

   (b) sum of squares $=\frac{157}{36}$, sum of reciprocals $=-\frac{5}{11}$

   (c) sum of squares $=-\frac{68}{49}$, sum of reciprocals $=\frac{2}{3}$

   ♦ The roots are not real, yet the sum of the squares and the sum of the reciprocals of these roots are. (The reason for this will be evident after you have studied complex numbers in Block D).

5. (a) $9x^2-43x+1=0$ (b) $27x^2-280x+1=0$ (c) $81x^2-1831x+1=0$

   (d) $3x^2-43x+3=0$

6. The method is an extension to that introduced in the solution to question 2.

   [Hint]: $(x-a)(x-b)(x-g)=x^3-a\,x^2-b\,x^2-g\,x^2+a\,b\,x+ag\,x+bg\,x-a\,bg$

7. (a) $u_n=5\cdot 3^n-2\cdot 4^n$, $n=0,1,2,\ldots$ (b) $u_n=11-3\cdot 2^n$, $n=0,1,2,\ldots$

   (c) $u_n=\frac{5}{13}\left((13-\sqrt{13})(3+\sqrt{13})^n+(13+\sqrt{13})(3-\sqrt{13})^n\right)$, $n=0,1,2,\ldots$

8. (a) $u_n=(5-4n)\cdot 5^{n-1}$, $n=0,1,2,\ldots$ (b) $u_n=(6-7n)\cdot 6^{n-1}$, $n=0,1,2,\ldots$

   (c) $u_n=2^{n+2}$, $n=0,1,2,\ldots$

9. (a) $F_{10}=55$ (b) $F_{20}=6765$ (c) $F_{30}=832040$ (d) $F_{40}=102334155$

   (e) $F_{50}=12586269025$

10. The range variable $n=0,1,2,\ldots$ used in the course materials creates an additional term $L_0=2$. This means that although each L_i are the same, their position in the sequence differs by 1. For example, with either range variable the term L_9 is 76. However, it is the ninth term of the sequence with range variable $n=1,2,3,\ldots$ but the 10th term of the sequence with range variable $n=0,1,2,\ldots$

11. (a) 1, 1, 2, 3, 5, 8, 13, 21, 34, 55

(b) We must firstly re-arrange $F_{n+2} = F_{n+1} + F_n$ as $F_{n+1} = F_{n+2} - F_n$. Then

$$
\begin{aligned}
F_2 &= F_3 - F_1 \\
F_4 &= F_5 - F_3 \\
F_6 &= F_7 - F_5 \\
\vdots\ & \quad \vdots \quad \vdots \quad \vdots \\
F_{2n} &= F_{2n+1} - F_{2n-1}
\end{aligned}
$$

Adding the LHS gives $F_2 + F_4 + F_6 + \ldots + F_{2n}$

And adding the RHS gives $F_{2n+1} - F_1 = F_{2n+1} - 1$ (as $F_1 = 1$).

Hence $F_2 + F_4 + F_6 + \ldots + F_{2n} = F_{2n+1} - 1$

(c) As f satisfies the golden ratio equation, we know that $f^2 = f + 1$. Squaring both sides gives $f^4 = f^2 + 2f + 1$, from which $f^4 - 1 = f^2 + 2f$. If we now divide both sides by f^2, the result $\dfrac{f^4 - 1}{f^2} = 1 + \dfrac{2}{f}$ follows immediately.

$$\therefore \frac{f^4 - 1}{f^2} = 1 + \frac{2}{\frac{1}{2}(1+\sqrt{5})} = 1 + \frac{4}{1+\sqrt{5}} = \sqrt{5} \quad \text{(after rationalising)}$$

Similarly, $\dfrac{y^4 - 1}{y^2} = 1 + \dfrac{2}{y} = 1 + \dfrac{2}{\frac{1}{2}(1-\sqrt{5})} = 1 + \dfrac{4}{1-\sqrt{5}} = -\sqrt{5}$

(d)

$$
\begin{aligned}
F_{n+1}^2 - F_{n-1}^2 &= \frac{1}{5}\left(f^{n+1} - y^{n+1}\right)^2 - \frac{1}{5}\left(f^{n-1} - y^{n-1}\right)^2 \\
&= \frac{1}{5}\left[f^{2n+2} - 2f^{n+1}y^{n+1} + y^{2n+2} - f^{2n-2} + 2f^{n-1}y^{n-1} - y^{2n-2}\right] \\
&= \frac{1}{5}\left[f^{2n+2} + y^{2n+2} - f^{2n-2} - y^{2n-2}\right] \quad \left(\text{As } f^{n+1}y^{n+1} = f^{n-1}y^{n-1} = (-1)^{n-1}\right) \\
&= \frac{1}{5}\left[f^{2n}f^2 - f^{2n}f^{-2} + y^{2n}y^2 - y^{2n}y^{-2}\right] \\
&= \frac{1}{5}\left[f^{2n}\left(f^2 - \frac{1}{f^2}\right) + y^{2n}\left(y^2 - \frac{1}{y^2}\right)\right] \\
&= \frac{1}{5}\left[f^{2n}\left(\frac{j^4 - 1}{f^2}\right) + y^{2n}\left(\frac{y^4 - 1}{y^2}\right)\right] \\
&= \frac{1}{5}\left[f^{2n} \cdot \sqrt{5} - y^{2n} \cdot \sqrt{5}\right] \\
&= \frac{1}{\sqrt{5}}\left[f^{2n} - y^{2n}\right] \\
&= F_{2n}
\end{aligned}
$$

12. $F_{n-1}+F_{n+1}=\frac{1}{\sqrt{5}}\left(f^{n-1}-y^{n-1}\right)+\frac{1}{\sqrt{5}}\left(f^{n+1}-y^{n+1}\right)$

$$=\frac{1}{\sqrt{5}}\left[f^{n}\cdot\frac{1}{f}-y^{n}\cdot\frac{1}{y}+f^{n}\cdot f-y^{n}\cdot y\right]$$

$$=\frac{1}{\sqrt{5}}\left[f^{n}\left(f+\frac{1}{f}\right)-y^{n}\left(y+\frac{1}{y}\right)\right]$$

$$=\frac{1}{\sqrt{5}}\left[\sqrt{5}\cdot f^{n}+\sqrt{5}\cdot y^{n}\right]$$

$$=f^{n}+y^{n}$$

$$=L_n$$

$$F_n\cdot L_n=\frac{1}{\sqrt{5}}\left(f^{n}-y^{n}\right)\cdot\left(f^{n}+y^{n}\right)$$

$$=\frac{1}{\sqrt{5}}\left[\left(f^{n}\right)^2-\left(y^{n}\right)^2\right]$$

$$=\frac{1}{\sqrt{5}}\left(f^{2n}-y^{2n}\right)$$

$$=F_{2n}$$

13. (a) $L_{2n}-L_n^2=\left(f^{2n}+y^{2n}\right)-\left(f^{n}+y^{n}\right)^2$

$$=f^{2n}+y^{2n}-\left(f^{2n}+2f^{n}y^{n}+y^{2n}\right)$$

$$=f^{2n}+y^{2n}-f^{2n}-2\left(fy\right)^{n}-y^{2n}$$

$$=-2\cdot\left(-1\right)^{n}$$

$$\therefore L_{2n}=L_n^2-2\left(-1\right)^{n}$$

(b) To show that $F_{3n}=F_n\cdot(L_{2n}+(-1)^n)$ it is sufficient to show that $F_{3n}-F_nL_{2n}=F_n\cdot(-1)^n$. We have

$$F_{3n}-F_n\cdot L_{2n}=\frac{1}{\sqrt{5}}\left(f^{3n}-y^{3n}\right)-\frac{1}{\sqrt{5}}\left(f^{n}-y^{n}\right)\left(f^{2n}+y^{2n}\right)$$

$$=\frac{1}{\sqrt{5}}\left[f^{3n}-y^{3n}-\left(f^{3n}+f^{n}y^{2n}-y^{n}f^{2n}-y^{3n}\right)\right]$$

$$=\frac{1}{\sqrt{5}}\left[y^{n}f^{2n}-f^{n}y^{2n}\right]$$

$$=\frac{1}{\sqrt{5}}\left[f^{n}\left(fy\right)^{n}-y^{n}\left(fy\right)^{n}\right]$$

$$=\frac{1}{\sqrt{5}}\left(f^{n}-y^{n}\right)\cdot\left(fy\right)^{n}$$

$$=F_n\cdot(-1)^n$$

$$\therefore F_{3n}=F_n\cdot(L_{2n}+(-1)^n)$$

14. $f^1 = \frac{1}{2}\left(1\cdot\sqrt{5}+1\right)$; $f^2 = \frac{1}{2}\left(1\cdot\sqrt{5}+3\right)$; $f^3 = \frac{1}{2}\left(2\cdot\sqrt{5}+4\right)$;

$f^4 = \frac{1}{2}\left(3\cdot\sqrt{5}+7\right)$; $f^5 = \frac{1}{2}\left(5\cdot\sqrt{5}+11\right)$; $f^6 = \frac{1}{2}\left(8\cdot\sqrt{5}+18\right)$;

$f^7 = \frac{1}{2}\left(13\cdot\sqrt{5}+29\right)$; $f^8 = \frac{1}{2}\left(21\cdot\sqrt{5}+47\right)$; $f^9 = \frac{1}{2}\left(34\cdot\sqrt{5}+76\right)$

The a_n are the Fibonacci numbers, and the b_n are the Lucas numbers. Hence

$$f^n = \frac{1}{2}\left(\sqrt{5}\cdot F_n + L_n\right)$$

15. $u_0 = 1$; $u_1 = 4$; $u_2 = 4$; $u_3 = -8$; $u_4 = -32$; $u_5 = -32$; $u_6 = 64$;
$u_7 = 256$; $u_8 = 256$; $u_9 = -512$; $u_{10} = -2048$

- The auxiliary equation is $l^2 - 2l + 4 = 0$. This has no real roots.
- The terms are the same
- The conjecture is that $u_n = 2^n\left(\cos\frac{n\pi}{3} + \sqrt{3}\sin\frac{n\pi}{3}\right), n = 0,1,...$ is the closed form for the given recurrence system. This may not be correct, however, because we cannot claim to have proved a general result following a finite number of trials.

16. (i) $u_2 = u_3 = u_4 = u_5 = 5$

(ii) $u_2 = \frac{3}{2},\ u_3 = \frac{3}{4},\ u_4 = \frac{3}{8},\ u_5 = \frac{3}{16}$

(iii) $u_2 = 64,\ u_3 = 256,\ u_4 = 1024,\ u_5 = 4096$

It appears that all three sequences are geometric; (i) with $r = 1$ (ii) with $r = \frac{1}{2}$ and (iii) with $r = 4$. In fact, we can easily see why:

$$u_{n+1} = \frac{u_n^2}{u_{n-1}} \Rightarrow u_{n+1} = \left(\frac{u_n}{u_{n-1}}\right)\cdot u_n. \text{ Hence } r = \frac{u_1}{u_0} \text{ and } u_n = \left(\frac{u_1}{u_0}\right)^{n-1}\cdot u_1,\ n = 0,1,2,....$$

The long-term behaviour of the sequence depends on both the relative sizes and signs of the two starting values u_0 and u_1, and is summarised in the table below:

u_0	u_1	Condition	Long term behaviour
positive	positive	$u_0 < u_1$	Diverges, to $+\infty$
positive	positive	$u_0 > u_1$	Converges $\to 0$ through positive values
negative	negative	$u_0 < u_1$	Converges $\to 0$ through negative values
negative	negative	$u_0 > u_1$	Diverges, to $-\infty$
positive	negative	$u_0 < \lvert u_1 \rvert$	Diverges and oscillates
positive	negative	$u_0 > \lvert u_1 \rvert$	Converges $\to 0$ (oscillating)
negative	positive	$\lvert u_0 \rvert < u_1$	Diverges and oscillates
negative	positive	$\lvert u_0 \rvert > u_1$	Converges $\to 0$ (oscillating)

17. (*i*) $u_2 = 43,\ u_3 = 321,\ u_4 = 2419,\ u_5 = 18217$

(*ii*) $A_2 = 8.6,\ A_3 \approx 7.4651,\ A_4 \approx 7.5358,\ A_5 \approx 7.5308$

(*iii*) $A_{n+1} = \dfrac{u_{n+1}}{u_n} = \dfrac{7u_n + 4u_{n-1}}{u_n} = 7 + \dfrac{4u_{n-1}}{u_n} = 7 + \dfrac{4}{A_n}$

(*iv*) $A = 7 + \dfrac{4}{A} \Rightarrow A^2 - 7A - 4 = 0$, from which $A = \dfrac{1}{2}\left(7 \pm \sqrt{65}\right)$

(*v*) We would expect the A_i to converge to $A = \dfrac{1}{2}\left(7 + \sqrt{65}\right)$

18. The closed form of a general arithmetic sequence with first term a and common difference d is $u_n = a + (n-1)d,\ \ n = 1,2,3,\ldots$.

Suppose that the second order recurrence relation is defined by $u_{n+1} = ku_n + lu_{n-1}$. Then substituting $u_n = a + (n-1)d$ into this recurrence relation gives

$$u_{n+1} = ku_n + lu_{n-1} \Rightarrow a + nd = k\left[a + (n-1)d\right] + l\left[a + (n-2)d\right]$$

$$\begin{aligned} \therefore\ a + nd &= k\left[a + nd - d\right] + l\left[a + nd - 2d\right] \\ &= ak + knd - kd + al + dnl - 2dl \\ &= \left[ak + al\right] + \left[knd + dnl\right] - \left[(k+2l)d\right] \\ &= a(k+l) + (k+l)nd - (k+2l)d \end{aligned}$$

In order that both sides of this equation are equivalent, we require that both $k + l = 1$ and $k + 2l = 0$ are both true.

There is only one solution to this pair of simultaneous equations, which is $k = 2$ and $l = -1$. Substituting these values into $u_{n+1} = ku_n + lu_{n-1}$, we see that the only recurrence relation whose solution generates an arithmetic sequence is $u_{n+1} = 2u_n - u_{n-1},\ n = 1,2,3,\ldots$